On some Family of Multivalent Functions.

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§ 1. Family $E_p$ of regular functions.

Let the function

$$F(z) = \sum_{n=0}^{p} a_n z^n.$$  

be regular in the unit-circle, where $p$ is any positive integer. [1]

We denote by $S_p$ or $K_p$ respectively the family of functions $F(z)$ by which the unit-circle is mapped into a star-like region with respect to the origin or a convex region, then the following theorems are well known. [2]

Theorem 1.

The necessary and sufficient condition that $F(z)$ should belong to the family $S_p$ is

$$R \left( \frac{z F'(z)}{F(z)} \right) > 0 \quad (|z| < 1).$$

Theorem 2.

The necessary and sufficient condition that $F(z)$ should belong to the family $K_p$ is

$$1 + R \left( \frac{z F''(z)}{F'(z)} \right) - \frac{p-1}{p} R \left( \frac{z F'(z)}{F(z)} \right) > 0 \quad (|z| < 1).$$

Now we denote by $E_p$ the family of functions $F(z)$ having the following two properties.

1° The mapped region of $|z| < 1$ by $F(z)$ is $p$-valent.

2° At any point of the mapped curve of $|z| = r$, where $r$ is any positive number less than 1, by $F(z)$, the curvature is positive and finite determinate.

And we call $F(z)$ a quasi-convex function. We can understand, by this definition, that a quasi-convex function $F(z)$ is convex on Riemann surface of $F(z)$. Then we have

Theorem 3.

The necessary and sufficient condition that $F(z)$ should belong to the family $E_p$ is

$$1 + R \left( \frac{z F''(z)}{F'(z)} \right) > 0 \quad (|z| < 1).$$

Proof: If

$$1 + R \left( \frac{z F''(z)}{F'(z)} \right) = R \left( \frac{z F'(z)}{F(z)} \right) > 0,$$

then

$$z F'(z) = p z^p + (p+1) a_{p+1} z^{p+1} + \ldots$$

and we have

$$\left. \frac{F'(z)}{z^{p-1}} \right|_{z=0} = p \neq 0.$$
Therefore \( \frac{F'(z)}{z^{p-1}} \) does not vanish in \( |z| < 1 \) and \( F'(z) \neq 0 \) in \( 0 < |z| < 1 \). (3)

Now we denote by \( \rho \) the curvature at any point on the mapped curve \( C \) of \( |z| = r \) \((0 < r < 1)\), then

\[
\rho = \frac{1}{zF'(z)} \left| 1 + \frac{zF''(z)}{F'(z)} \right| > 0,
\]

which is the property 2°.

As \( F'(z) \neq 0 \) on \( |z| = r \), \( C \) is a regular curve and the angle from the real-axis to the tangent line at any point on \( C \) is given by \( \arg izF'(z) \). So that we have, as \( z \) describes on \( |z| = r \),

\[
\int \text{darg}izF'(z) = \int \text{darg}Z^p + \int \text{darg} \frac{F'(z)}{Z^{p-1}} = \int \text{darg}Z^p = 2p\pi,
\]

where \( \frac{F'(z)}{Z^{p-1}} \) does not vanish, as cited above.

Therefore the mapped curve \( C \) is closed and \( p \)-valent, and, being \( r \) arbitrary, the mapped region of \( |z| < 1 \) is \( p \)-valent which proves the property 1°.

Conversely, if we have properties 1°, 2°, then the curvature

\[
\rho = \frac{1}{zF'(z)} \left| 1 + \frac{zF''(z)}{F'(z)} \right|
\]

at any point \( z \) on \( |z| = r \) \((0 < r < 1)\) is positive, and, being \( r \) arbitrary, we have

\[
1 + R \left| \frac{zF''(z)}{F'(z)} \right| > 0 \quad (|z| < 1).
\]

Thus our theorem is completely proved.

§ 2. Relations among \( S_p \), \( K_p \), and \( E_p \).

If \( F(z) \) belongs to any one of \( S_p \), \( K_p \) and \( E_p \), then \( F(z) \) is \( p \)-valent in \( |z| < 1 \), and consequently \( F(z) \) does not vanish in \( 0 < |z| < 1 \). Hence there exists the regular function \( h(z) \) such that

\[
h(z) = z^p \sqrt[1/p]{\frac{F(z)}{Z^p}}, \quad F(z) = [h(z)]^p, \quad h(o) = 0, \quad h'(o) = 1 \quad (|z| < 1),
\]

and we have, for \( h(z) \), the relations

\[
R \left[ z \frac{F'(z)}{F(z)} \right] = pR \left[ z \frac{h'(z)}{h(z)} \right]
\]

\[
(2.1) \quad 1 + R \left[ z \frac{F''(z)}{F'(z)} \right] = 1 + R \left[ z \frac{h''(z)}{h'(z)} \right] + (p-1) R \left[ z \frac{h'(z)}{h(z)} \right]
\]

\[
1 + R \left[ z \frac{F''(z)}{F'(z)} \right] - \frac{p-1}{p} R \left[ z \frac{F'(z)}{F(z)} \right] = 1 + R \left[ z \frac{h'(z)}{h'(z)} \right].
\]

We get immediately, from these relations, the following

Theorem 4.

Suppose that \( F(z) \in S_p \), then \( h(z) \in S_1 \) and suppose that \( F(z) \in K_p \), then \( h(z) \in K_1 \).

If we assume that \( F(z) \in K_p \) and therefore \( h(z) \in K_1 \), then

\[
R \left[ z \frac{h'(z)}{h(z)} \right] > \frac{1}{2}
\]
by the theorem due to M. Strohacker [4], and, from (2.1),

\[
1 + R \left( \frac{z F''(z)}{F'(z)} \right) > \frac{1}{2} (p-1) \geq 0, \quad R \left( \frac{z F'(z)}{F(z)} \right) > \frac{p}{2}.
\]

This proves the following

**Theorem 5.**

*Suppose that \( F(z) \in K_p \), then \( F(z) \in E_p \) and \( F(z) \in S_p \).*

Next we assume that \( E(z) \in E_p \) and denote by \( \varphi \) the angle from the real-axis to the tangent line at \( F(z) \), where \( Z = re^{i\theta} \), on the mapped curve of \( |z| = r \), then

\[
\frac{d\varphi}{d\theta} = \frac{d}{d\theta} \arg z F'(z) = 1 + R \left( \frac{z F''(z)}{F'(z)} \right) > 0.
\]

So that the tangent line rotates such that \( \varphi \) increases as \( Z \) describes in positive sense on \( |z| = r \). And, as the curvature \( \rho \) is positive, the radius vector \( F(z) \) rotates in positive sense. Hence we have

\[
R \left( \frac{z F'(z)}{F(z)} \right) = \frac{d}{d\theta} \arg F(z) > 0.
\]

which reduces to the following theorem.

**Theorem 6.**

*Suppose that \( F(z) \in E_p \), then \( F(z) \in S_p \).*

We can represent the theorem 5 and 6 symbolically by

\[
K_p \subset E_p \subset S_p, \quad K_1 = E_1 \subset S_1
\]

by (2.1).

Lastly we have the theorem as follows, from the relation

\[
R \left( \frac{z (z F'(z))^p}{z F'(z)} \right) = 1 + R \left( \frac{z F''(z)}{F'(z)} \right).
\]

**Theorem 7.**

*If \( F(z) \in E_p \), then \( \frac{1}{p} z F'(z) \in S_p \) and if \( F(z) \in S_p \), then

\[
p \int \frac{F(z)}{z} \, dz \in E_p.
\]

§ 3. The circle of quasi-convexity for the functions \( \subset S_p \).

Suppose that \( F(z) \in S_p \), then

\[
R \left( \frac{z F'(z)}{F(z)} \right) > 0,
\]

and

\[
\left[ \frac{z F'(z)}{F(z)} \right]_{z=0} = p.
\]

Therefore,

\[
p \frac{1-r}{1+r} \leq R \left( \frac{z F'(z)}{F(z)} \right) \leq p \frac{1+r}{1-r} \quad (|z| \leq r),
\]

and we have, by the well known theorem of G. Julia,
Combining these inequalities, we get

\[ 1 + R \left( z \frac{F''(z)}{F'(z)} \right) \geq \frac{pr^2 - 2(p+1)r + p}{1 - r^2} \quad (|z| \leq r), \]

and hence, for \(|z| < \sigma_p = \frac{1}{p}(p+1 - \sqrt{2p+1}),\)

\[ 1 + R \left( z \frac{F''(z)}{F'(z)} \right) > 0, \]

and that, for the function

\[ F(z) = \frac{z^n}{(1-z)^{2p}} \in S_p, \]

we have

\[ 1 + R \left( z \frac{F''(z)}{F'(z)} \right) = 0 \quad (Z = -\sigma) \]

Thus we conclude the following

**Theorem 8.**

Suppose that \( F(z) \in S_p \), then \( F(z) \in E_p \) for \(|z| < \sigma_p = \frac{1}{p}(p+1 - \sqrt{2p+1}).\)

And \( \sigma_p \) is the greatest number for quasi-convexity.

§ 4. Distortion theorem and Coefficient problem for \( E_p \).

For the function \( F(z) \in E_p \), we have, by the theorem 7,

\[ \frac{1}{p} z F'(z) \in S_p, \]

and consequently,

\[ \frac{|z|^p}{(1+|z|)^{2p}} \leq \frac{1}{p} |z| F'(z) \leq \frac{|z|^p}{(1-|z|)^{2p}}. \]

From this relation, we can prove the following theorem by the analogous method for the case \( p = 1 \).

**Theorem 9.**

Let \( F(z) = \sum_{n=P} a_n z^n \ (a_P = 1) \in E_p \), then

\[ \frac{p|z|^{p-1}}{(1+|z|)^{2p}} \leq |F'(z)| \leq \frac{p|z|^{p-1}}{(1-|z|)^{2p}} \]

\[ p \int_0^{|z|} \frac{Z^{p-1}}{(1+Z)^{2p}} dZ \leq |F(z)| \leq p \int_0^{|z|} \frac{Z^{p-1}}{(1-Z)^{2p}} dZ. \]

And the equality signs are true for

\[ F(z) = p \int_0^z \frac{Z^{p-1}}{(1-Z)^{2p}} dZ \in E_p. \]

We have already known, for \( F(z) \in S_p \), that
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\[ |a_{p+1}| \leq \frac{2p(2p+1)\ldots(2p+k-1)}{k!}, \quad K = 1, 2, \ldots. \]

and the equality holds for \( F(z) = \frac{z^p}{(1-z)^{2p}} \). \[5\]

We can represent this theorem, by Majorant symbol, as follows,

\[ F(z) \ll \frac{z^p}{(1-z)^{2p}}. \]

Now, if \( F(z) \in E, \) then \( \frac{1}{p} z F'(z) \in S_\alpha \) and therefore

\[ \frac{1}{p} z F'(z) \ll \frac{z^p}{(1-z)^{2p}}. \]

From this relation, we have

\[ F(z) \ll p \int_0^z \frac{z^{p-1}}{(1-z)^p} \, dz \]

i.e.

\[ F(z) \ll p \int_0^z \left[ z^{p-1} + \sum_{k=1}^\infty \frac{2p(2p+1)\ldots(2p+k-1)}{k!} z^{p+k-1} \right] \, dz \]

Consequently the following result is obtained.

**Theorem 10.**

Let \( F(z) = \sum_{n=p} a_n z^n \quad (a_p = 1) \in E_p, \) then

\[ |a_{p+1}| \leq \frac{2p(2p+1)\ldots(2p+k-1)}{k!(p+k)} \quad (k = 1, 2, \ldots) \]

and equality holds true for the function

\[ F(z) = p \int_0^z \frac{Z^{p-1}}{(1-Z)^{2p}} \, dZ. \]

§ 5. Some lemmas.

In this section, some inequalities, all of which hold for \( |z| \leq r \), will be given for the preparation of next section.

Let \( F(z) \) be bounded \( (|F(z)| < M) \), then we have

**Lemma 1.**

\[ m_{r_1}^{p_1} \frac{1-Mr}{M-r} \leq |F(z)| \leq Mr_1^{p_1} \frac{1+Mr}{M+r}. \]

Now putting \( \varphi(z) = \frac{F(z)}{Z^p} \), then \( |\varphi(z)| \leq M, \varphi(0) = 1 \) and therefore

\[ |\varphi'(z)| \leq \frac{M^p - |\varphi(z)|^2}{M(1-r^2)}. \]

i.e.,

\[ (5.1) \quad \left| \frac{F'(z)}{Z^p} - p \frac{F(z)}{Z^{p+1}} \right| \leq \frac{M^{p+1} - |F(z)|^2}{M^{p+1}(1-r^2)}. \]
Combining (5,1) with lemma 1, we have

**Lemma 2.**

\[
|F'(z)| \geq Mr^{p-1} \frac{p(\gamma)}{(M-\gamma)^2},
\]

where \( p(\gamma) = pM - ((p+1)M^2 + (p-1)\gamma + pMr^2). \)

and

**Lemma 3.**

\[
|\phi(z)| \geq \frac{Mr^p}{(M-\gamma)^2} (1-2M\gamma + \gamma^2),
\]

where \( \phi(z) = zF'(z) - (p-1) F(z). \)

From Lemma 1, 2, (5,1), we get

**Lemma 4.**

\[
\nu(z) \geq \frac{M^p r^{2(p-1)}}{(M-\gamma)^2} (1-2M\gamma + \gamma^2) (M-2\gamma + M\gamma^2)
\]

,where \( \nu(z) = M^p r^{3(p-1)} |F(z)|^2. \)

and

**Lemma 5.**

\[
|\phi(z)| \geq \frac{R(\gamma(z))}{zF'(z)} \geq \frac{M^{p-1} - 2M\gamma + \gamma^2}{P(\gamma)}.
\]

Next we have, applying (5,1) and lemma 1 to

\[
\frac{|\phi(z)|}{M^p + |F(z)|} = \frac{|F(z)|}{M^p + |F(z)|} \left| \frac{Z F'(z) - (p+1)}{F(z)} \right|
\]

**Lemma 6.**

\[
\frac{|\phi(z)|}{M^p + |F(z)|} \geq \frac{\gamma(1-2M\gamma + \gamma^2)}{(1-\gamma^2)(M-\gamma)}.
\]

Lastly lemma 2 and 4 conclude

**Lemma 7.**

\[
\frac{\nu(z)}{M(1-\gamma^2)^{p-2} |F'(z)|} \leq \frac{M^p (M-2\gamma + M\gamma^2)}{(1-\gamma^2) P(\gamma)}
\]

The equality sign in lemma 1 ~ 7 holds at \( z=\gamma \) for the function

\[
F(z) = Mz^p \frac{1-Mz}{M-z}.
\]

§ 6. The circle of quasi-convexity for the bounded functions.

Let \( F(z) \) be bounded \((|F(z)| < M)\), then the author has proved [6] that the circle of convexity of \( F(z) \) for \( K_t = E_t \) is given by \( |z| < \rho_t \), where \( \rho_t \) is the positive root less than 1 of the equation

\[
M - (4M^2 - 1) Z + 3MZ^2 - Z^3 = 0.
\]

Now we shall generalise this theorem for the functions of \( F_z \).

If we define the function \( \phi(z) \) regular in \( |z| < 1 \) by

\[
\phi(z) = M^p \frac{z^{p-1} F(w) - w^{p-1} F(z)}{M^p z^{p-1} w^{p-1} - F(z) F(w)}, \quad W = \frac{-S + z}{1 - z S}, \quad (|z| < 1),
\]
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then \( |\phi(s)| < M \), \( \phi(0) = 0 \) and, by the theorem of bounded functions, \[ M - \frac{|\phi'(0)|}{M} \geq \frac{1}{2} |\phi''(0)|. \]

This inequality is reduced to

\[
(6.1) \quad R \left[ z \frac{F''(z)}{F'(z)} \right] \geq (p-1)(p-2) R \left[ z \frac{F(z)}{zF'(z)} \right] + \frac{2}{1-|z|^3} \frac{z}{R} \left[ F'(z) \right] + 2M \frac{\lambda(z)}{zF(z)} |z|^{-2} \]

\[ + 2R \left[ (p-1)M |z|^{2(p-1)-1} - zF(z)F'(z) \right] \cdot \frac{\lambda(z)}{zF'(z)} - 2 \frac{\nu(z)}{M(1-|z|^3)} |z|^{-2} \frac{F'(z)}{F(z)}, \]

where \( F'(z) \neq 0 \) and \( \lambda(z) = zF'(z) - (p-1)F(z) \), \( \nu(z) = M^2 |z|^{2(p-1)-1} > |F(z)|^2. \)

The fourth term of the right-hand side in (6.1) is reduced to

\[ 2 R \left[ (p-1) \frac{F(z)}{zF'(z)} \right] \lambda(z) \frac{\nu(z)}{zF'(z)}, \]

which is not less than

\[ 2 (p-1) R \left[ \lambda(z) \frac{\nu(z)}{zF'(z)} \right] - 2 \left| \frac{F(z)}{\nu(z)} \right| \left| \frac{\lambda(z)}{zF'(z)} \right| \]

and

\[ (p-1) \frac{F(z)}{zF'(z)} = 1 - \frac{\lambda(z)}{zF'(z)}. \]

Therefore we have

\[
(6.2) \quad R \left[ z \frac{F''(z)}{F'(z)} \right] \geq p-2 + \frac{2}{1-|z|^3} + p \frac{\lambda(z)}{zF'(z)} + 2 \left| \frac{\lambda(z)}{zF'(z)} \right| \frac{\nu(z)}{zF'(z)} + 2M \frac{\lambda(z)}{zF(z)} |z|^{-2} \frac{F'(z)}{F(z)} \]

\[ - 2 \frac{\nu(z)}{M(1-|z|^3)} |z|^{-2} \frac{F'(z)}{F(z)}. \]

Applying Lemma 1 \( \sim 7 \) in § 5, the inequality

\[
(6.3) \quad 1 + R \left[ Z \frac{F''(z)}{F'(z)} \right] \geq \frac{K(\gamma)}{P(\gamma)(M-\gamma)} \]

holds for \( |z| \leq \gamma \), where

\[
(6.4) \quad K(\gamma) = M^p \gamma^2 - M \{(p+1)M^2 + 2p^2 - 2p - 1\} \gamma + \{(2p^2 + 2p - 1)M^2 + (p-1)^2\}\gamma^2 - M \gamma^2. \]

The equality sign in (6.3) holds at \( z = \gamma \) for the function

\[ F(z) = \frac{Mz^\gamma}{M-z}. \]

As it is easily proved that the equation \( K(\gamma) = 0 \) has only one positive root \( \rho_p \) less than \( 1 \) and \( K(\gamma) \) and \( P(\gamma) \) and positive for \( 0 < \gamma < \rho_p \), we have the result as follows.

**Theorem 11.**

Let \( F(z) = \sum_{n=p}^\infty a_n z^n \) be regular and bounded \( (|F(z)| < M) \)

in \( |z| < 1 \), then \( F(z) \) is quasi-convex in \( |z| < \rho_p \), where \( \rho_p \) is a positive root less than \( 1 \) of the equation \( K(\gamma) = 0 \). And \( \rho_p \) is the possible number for quasi-convexity.
If we assume that \( p \) is any real number, then we can prove
\[
\frac{d\rho_p}{dp} > 0,
\]
so that \( \rho_p \) increases with \( p \) and hence
\[
\rho_1 < \rho_2 < \ldots < \rho_p < \rho_{p+1} < \ldots.
\]
Consequently we have the following corollary of theorem 11.

Corollary.

Let \( F(z) \) be any function in theorem 11, then \( F(z) \) is quasi-convex
in \( |z| < \rho_1 \) for all positive integer \( p \), where \( \rho_1 \) is a positive root less
than 1 of the equation
\[
M - (4M^2 - 1)z + 3MZ^2 - Z^3 = 0.
\]

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References.

[1] In this note, the function \( F(z) \) is always of the form \((1.1)\).


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