How Many Live Minimal Structural Traps Are Required to Make a Minimal Deadlock Locally Live in General Petri Nets?

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Petri nets are one of useful models for discrete event systems in which liveness problem as well as reachability problem is one of big issues. But, it has not been completely solved from the point of view of useful initial-marking-based liveness conditions in general Petri nets. In this paper, to guarantee local liveness (i.e., liveness under Mod) for each minimal deadlock (MSDL), \(MD = (SD, TD, FD, M_{d})\), with real deadlock-trap structure, it is shown that the minimum number of required live minimal structural traps (MSTRs), \(NT = (ST, TT, FT, M_{t})\) s.t. \(SD \supseteq ST\), is conditionally (which means that the conditions of Lemma 4-9 are fulfilled for a bounded MSDL \(MD\) containing at least one MSTR \(NT\) s.t. \(SD \supseteq ST\) and see also Remarks 4-2 (3) in Subsection 4.3) "one". Note that this local liveness for \(MD\) s.t. \(SD \supseteq ST\) is one of useful necessary conditions for liveness condition of general Petri nets \(N = (S, T, F, M)\) s.t. \(S \supseteq S_{o}\). However, because this has not been discussed in literature and is not trivial, some new concepts such as \(T\)-cornucopias and return paths are introduced into the real deadlock-trap structure s.t. \(SD \supseteq ST\) in \(N\) and this is proven by dividing it into two cases: \(ND\) s.t. \(SD \supseteq ST\) is live and unbounded under \(M_{d}\) and \(ND\) s.t. \(SD \supseteq ST\) is live and bounded under \(M_{d}\). Moreover, another related problem is also discussed. Usefulness for the results obtained is also discussed.

1 Introduction

It is widely recognized that Petri nets are one of useful models for discrete event systems \([1\sim 3]\). Although liveness problem on general Petri nets is one of big issues, it has not been completely solved from the point of view of useful initial-marking-based liveness conditions \([4\sim 7]\). However, it is evident that one of useful necessary conditions for liveness on general Petri nets \(N = (S, T, F, M)\) is each minimal deadlock (MSDL) \(ND = (SD, TD, FD, M_{d})\) to be locally live in the sense of (iii) in Subsection 3.1 (see Lemma 3-1 and Remarks 3-1 (3)). Moreover, if we consider only MSDL \(ND\) s.t. \(SD \supseteq ST\) i.e., real deadlock-trap structure (real d-t structure for short) in \(N\), we can make \(ND\) s.t. \(SD \supseteq ST\) live under \(M_{d}\) because of some live minimal structural traps (MSTRs), \(NT = (ST, TT, FT, M_{t})\) s.t. \(SD \supseteq ST\), where we call this kind of liveness of \(ND\) as "real d-t properties" hereafter.

In this paper, in order to guarantee local liveness for MSDL \(ND\) s.t. \(SD \supseteq ST\) in general Petri nets \(N\) (i.e., \(ND\) with real d-t structure) as the first step to the final goal of liveness on \(N\), we will consider the minimum number of required live MSTRs as in Problem 1 in Subsection 3.2 (see also Lemma 3-2 and Remarks 3-2). Although the answer to Problem 1 is simple and "one" conditionally (which means that the conditions of Lemma 4-9 are fulfilled for a bounded MSDL \(ND\) containing at least one MSTR \(NT\) s.t. \(SD \supseteq ST\) and see also Remarks 4-2 (3) in Subsection 4.3.), the reasons why it is not trivial \([8, 9]\). Then we will introduce some new concepts such as \(T\)-cornucopias and return paths in order to prove the above and to give an answer to Problem 1. Moreover, another related problem (i.e., Problem 2) is also discussed.

In Section 2, we will give some definitions and notations for general Petri nets \(N\). In Section 3, we will describe two problems and the backgrounds of this paper. In Section 4, we will give and prove the answer to

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Problem 1. Finally, we will present a sufficient condition for Problem 2 in Section 5.

2 Notations and Definitions

(1) A general Petri net is a 4-tuple, \( N = (S, T, F, M_o) \), where \( S \) is a finite set of places, \( T \) is a finite set of transitions, \( F \subseteq (S \times T) \cup (T \times S) \) is a set of arcs, and \( M_o \) is the initial marking under the conditions of \( S \cap T = \phi \) and \( S \cup T \neq \phi \).

(2) Let \((p, t) \in F \cap (S \times T)\) \((t, p) \in F \cap (T \times S)\), resp.) be an arc from \( p \in S \) \((t \in T, \text{resp.})\) to \( t \in T \) \((p \in S, \text{resp.})\). The arc \((p, t) \in F \) \(((t, p) \in F, \text{resp.})\) is called an input arc of \( t \) or an output arc of \( p \) (an output arc of \( t \) or an input arc of \( p \), resp.). One pair of arcs of \((p, t) \in F \) and \((t, p) \in F \) is called a "double arc" and is denoted \(-\ast\). For \( x \in S \cup T \), let \(*x\) and \( x*\) be defined as \(*x = \{ y | y \in S \cup T, (y, x) \in F \} \) and \( x* = \{ y | y \in S \cup T, (x, y) \in F \} \), respectively. Moreover \(*X \triangleq \ast X \) and \(*X \triangleq \ast X*\).

(3) In this paper, it is assumed that \( N \) is an ordinary Petri net whose arc weights are each one \([3]\).

(4) The changes of the marking occur due to the firing of a transition. The transition may fire when each of its input places contains at least one token. Transition firing results in removing the tokens from their input places and adding them to the output places.

(5) Let \( R(M_o) \) be the set of all markings reachable from \( M_o \). For a transition \( t \in T \) (a place \( p \in S \), resp.), \( t \) (\( p \), resp.) is said to be live under \( M_o \) if there exists \( M' \in R(M) \) such that \( t \) is fireable (\( p \) has at least one token, resp.) at \( M' \) for any marking \( M \in R(M_o) \). If each \( t \in T \) (each \( p \in S \), resp.) is live under \( M_o \), \( N \) is transition-live, t-live or TL (place-live, p-live or PL, resp.) under \( M_o \). The t-liveness used in this paper is equal to the most general liveness of level 4 of Ref.[3]. Moreover, note that each transition \( t \in T \) in \( N \) can fire selectively even if the firing condition for \( t \) is satisfied \([3]\). On the other hand, a transition \( t \in T \) in \( N \) is said to be dead if \( t \) can never fire in any firing sequence in the set of all possible firing sequences from \( M' \in R(M), M \in R(M_o) \). If \( N \) has at least one dead transition, \( N \) is said to be dead under \( M_o \).

(6) For two sets, \( X \) and \( Y \), \( X \setminus Y = \{ z | z \in X, z \notin Y \} \) and \(|X| \) means the number of elements of \( X \).

(7) Let \( N = (S,T,F,M) \) be a general Petri net.

(i) A nonempty set \( \hat{S}_Z \subseteq S \) is called a structural deadlock (or a siphon) iff \( \phi \subseteq^* \hat{S}_Z \subseteq \hat{S}_Z \).

(ii) A nonempty set \( \hat{S}_Z \subseteq S \) is called a structural trap iff \( \phi \subseteq^* \hat{S}_Z \subseteq^* \hat{S}_Z \).

(iii) Let \( \hat{S}_Z \) be a structural deadlock (trap, resp.). \( \hat{S}_Z \) is called minimal if there is no structural deadlock (trap, resp.) contained in \( \hat{S}_Z \) as a real subset. \( \hat{S}_Z \) is the maximal set which satisfies (i) (ii), resp.) and the net defined by \( \hat{S}_Z \) and \( \hat{S}_Z^* \) (\( ^* \hat{S}_Z \), resp.) is connected.

In this paper, note that the above maximality is defined in a restricted form than usual ones \([1-3]\) because it is more useful for liveness analysis aspects.

Structural deadlocks and traps are both special sets of places. Their important properties are that if a structural deadlock is empty under some marking then it remains empty under each successor marking (note that this can be considered as the general definition of deadlocks. See also Property 2-1 (i)), while if a structural trap is marked under some marking then it remains marked under each successor marking (note that this can be considered as the general definition of traps. See also Property 2-1 (ii)). \([1-3]\).

(8) We have a property for deadlocks and traps.

[Property 2-1] \([6,7]\). (i) Deadlocks in \( N \) are defined by only (7)-(i).

(ii) There exist two types of traps in connection with making a deadlock t-live: (a) one is (7)-(ii), and (b) the other is a behavioral trap which does not satisfy \( \phi \subseteq^* \hat{S}_Z \subseteq \hat{S}_Z \). See also Ref.[7].

(9) The following notations for deadlocks and traps in \( N \) are used in this paper because liveness depends on both structure and token distribution.

(i) A maximal (minimal, resp.) structural deadlock in \( N \) is denoted as SDL \( \hat{N}_D = (\hat{S}_D, \hat{T}_D, \hat{F}_D, \hat{M}_oD) \) (MSDL \( N_D = (S_D, T_D, F_D, M_oD) \), resp.) which is a subnet defined by \( \hat{S}_D \) (\( S_D \), resp.) s.t. \( \hat{T}_D = \hat{S}_D \) (\( T_D = S_D \), resp.).

(ii) A maximal (minimal, resp.) structural trap in \( N \) is denoted as STR \( \hat{N}_T = (\hat{S}_T, \hat{T}_T, \hat{F}_T, \hat{M}_oT) \) (MSTR \( N_T = (S_T, T_T, F_T, M_oT) \), resp.) which is a subnet defined by \( \hat{S}_T \) (\( S_T \), resp.) s.t. \( \hat{T}_T = \hat{S}_T \) (\( T_T = \hat{S}_T \), resp.).

(iii) A maximal (minimal, resp.) behavioral trap between \( t_a \in T_D \) and \( t_b \in T_D \) in \( N \) is denoted as BTR \( \hat{N}_{BT} = (\hat{S}_{BT}, \hat{T}_{BT}, \hat{F}_{BT}, \hat{M}_{oBT}) \) (MBTR \( N_{BT} = (S_{BT}, T_{BT}, F_{BT}, M_{oBT}) \), resp.) which is a subnet defined by \( \hat{S}_{BT} \) (\( S_{BT} \), resp.). Let the subnet defined by \( \hat{S}_{DB} = S_D \cup \hat{S}_{BT} \) denote \( N_{DB} = (S_{DB}, T_{DB}, F_{DB}, M_{oDB}) \). See also Refs.[7,10].

(10) A general Petri net \( N = (S,T,F,M) \) is said to be k-bounded or simply bounded (B) if the number of tokens in each place does not exceed a finite number \( k \) for any marking reachable from \( M_o \), i.e., \( M(p) \leq k \) for every place \( p \in S \) and every marking \( M \in R(M_o) \), where \( M(p) \) is the token count on \( p \) at the marking \( M \in R(M_o) \).

(11) Let \( N : X (N: X, \text{resp}) \) mean that a general Petri net \( N \) satisfies a property \( X \) (does not satisfy a property \( X \), resp.). Moreover, for example, let \( N : \bar{X} \cap Y \) mean that \( N \) does not satisfy the properties \( X \) and \( Y \).
In this paper, \( X = TL, PL, B, CT \) or \( AC_T \) is used (see also Definition 4-1).

(12) We use the symbol \( \supseteq \) to stress as follows in this paper: \( A \supseteq B \) means that \( B \) is contained in \( A \) as a real subset, where \( \supseteq \) equals \( \supset \). Let \( C \) and \( D \) be two statements. In this paper, \( C \rightarrow D \) means that if \( C \) is true, then \( D \) is also true.

(13) Due to Property 2-1 (i), we call a structural deadlock a deadlock in this paper.

3 Backgrounds and Problems of This Paper

3.1 Backgrounds of This Paper

In this subsection, we will consider liveness problem for general Petri nets. First, let us consider the following five assertions.

A(i); A general Petri net \( N = (S, T, F, M_o) \) is TL, B or \( \overline{\mathbb{B}} \), under \( M_o \).

A(ii) (A(iii), resp.): In \( N \), each SDL \( \tilde{N}_D = (S_D, T_D, F_D, M_{oD}) \) st. \( S \supseteq \tilde{S}_D \) is TL, B or \( \overline{\mathbb{B}} \), under \( M_{oD} \) (\( M_o \), resp.).

A(iii)(A(iii), resp.); In \( N \), each MSDL \( N_D = (S_D, T_D, F_D, M_{oD}) \) st. \( S \supseteq \tilde{S}_D \supseteq S_D \) is TL, B or \( \overline{\mathbb{B}} \), under \( M_{oD} \) or \( M_{oD}^{TB} \) (\( M_o \), resp.), where note that \( M_{oD} \) is an initial marking on \( S_D \supseteq S_T \) and \( M_{oD}^{TB} \) is that on \( S_D = S_D \cup \tilde{S}_D \) in which \( N_D, S_D \notin S_T, S_D \notin S_T \) or \( S_D \supseteq S_T \) (but \( N_D, S_D \supseteq S_T \) is dead due to violating real d-t properties) is activated by BTR \( \tilde{N}_B \).

Note that A(iii) (A(iii), resp.) means that every \( t \) in \( T_D \) (\( t \) in \( T_D \), resp.) of \( \tilde{N}_D (N_D, \text{resp.}) \) is TL under \( M_o \). Then, we have the next property.

**Lemma 3-1** Consider only general Petri nets \( N \) with at least one SDL \( \tilde{N}_D \) s.t. \( S \supseteq \tilde{S}_D \supseteq S_D \). Then we have A(i) \( \rightarrow \) A(ii) \( \rightarrow \) A(iii).

(Proof) The details for this proof have been given in Ref.[4]. Then we will give a brief description for this proof here. First, it is evident that A(i) \( \rightarrow \) A(ii) \( \rightarrow \) A(iii) is true because of \( S \supseteq \tilde{S}_D \supseteq S_D \) and the definition of t-liveness in general Petri nets \( N \), in other words, A(i) (A(iii), resp.) means that every \( t \in T_D (t \in T_D, \text{resp.}) \) of SDL \( \tilde{N}_D \) (MSDL \( N_D, \text{resp.}) \) in \( N \) is TL under \( M_o \).

Secondly, let us prove A(i) \( \rightarrow \) A(ii). \( \text{(1)} \) It is true that no token can be fed to the net \( \tilde{N}_D \) from the net of \( S \backslash \tilde{S}_D \) and the token movement on \( \tilde{S}_D \) is just controlled and restricted by the net of \( S \backslash \tilde{S}_D \) because of \( \tilde{S}_D \subseteq \tilde{S}_D^* \) and \( S_D \backslash \tilde{S}_D^* = \phi \). \( \text{(2)} \) If we assume that \( \tilde{N}_D \) is dead under \( M_{oD} \) and if we take \( \text{(1)} \) into consideration, \( \tilde{N}_D \) is also dead under \( M_o \). But, this means that \( N \) is not TL under \( M_o \), i.e., A(i) is false. \( \text{(3)} \) Therefore it is true that A(i) \( \rightarrow \) A(ii). Note that A(ii) \( \rightarrow \) A(iii) is also true because of the same reasons as those in the above proof for A(i) \( \rightarrow \) A(ii). See also Ref.[4], but note Remarks 3-1 (3) of this paper.

Q.E.D.

**Remarks 3-1** (1) Note that A(ii) (A(iii), resp.) is a useful necessary condition for A(i) (A(ii), resp.) as in Lemma 3-1.

(2) In Ref.[4], it is discussed that A(iii) \( \rightarrow \) A(ii) is not always true and that A(ii) \( \rightarrow \) A(i) is always true.

(3) If A(ii) (A(iii), resp.) is true, let \( \tilde{N}_D (N_D, \text{resp.}) \) be called locally live. Moreover, A(ii) (A(iii), resp.) is global liveness for \( N_D (N_D, \text{resp.}) \). Then it is evident that A(ii) (A(iii), resp.) is a useful necessary condition for A(ii) (A(iii), resp.). However, note the next; Since \( N_D \) s.t. \( S_D \subseteq S_T, S_D \notin S_T \) or \( S_D \supseteq S_T \) (but \( N_D \) s.t. \( S_D \supseteq S_T \) is dead under \( M_oD \) owing to violating real d-t properties) strictly is locally dead under \( M_oD \) in A(iii), we for simplicity use the extended local liveness for the above \( N_D \) as follows: \( N_D \) s.t. \( S_D \subseteq S_T, S_D \notin S_T \) or \( S_D \supseteq S_T \) (but dead under \( M_oD \)) is called locally live if \( N_D, S_D \notin S_T \) or \( S_D \supseteq S_T \) is live under \( M_oD \). Note that \( N_D \) of the above strictly is locally dead under \( M_oD \) and globally live under \( M_oD'B \), while \( N_D'B \) strictly is locally live under \( M_oD'B \). See also Refs.[6,7,10].

(4) In Ref.[4], for general Petri nets \( N \) with at least one SDL \( \tilde{N}_D \), the condition for A(ii) \( \supseteq \) A(iii) has been shown and the fact of A(i) \( \supseteq \) A(ii) \( \supseteq \) A(iii) has been proven.

(5) Note that the condition for A(iii) \( \supseteq \) A(iii) and some basic properties for A(iii) are discussed in Ref.[5].

(6) In Ref.[6], the necessary and sufficient condition for A(iii) of \( N_D \) s.t. \( S_D \supseteq S_T \) is obtained (i.e., real d-t properties) under the condition that no behavioral traps are used.

(7) In Ref.[7], the necessary and sufficient condition for A(iii) of \( N_D \) s.t. \( S_D \supseteq S_T \) (but, this \( N_D \) is dead owing to violating real d-t properties), \( S_D \notin S_T \) or \( S_D \notin S_T \) has been also obtained (i.e., virtual d-t properties owing to a behavioral trap (BTR)).

(8) In this paper, we want to give the basis for (6) directly or for (7) indirectly (see also Subsection 3.2).
That every N D i.e., 3t the initial marking in assertions A(iv) and A(v) is also to consider localliveness for the target MSTR NT. A{iii)-A{iv)-A(v). NT.

3.2 Problems of This Paper

Because of (a), we can assume without loss of generality that each MSDL under the next premisesj (a) Neglect the net of contained by each MSDL Sm.

[Remarks 3-2] For Problem 1, note that local liveness on MSDL is discussed under the next premises; (a) Neglect the net of S SD in other words, the net of S S D is TL under M D. (b) Because of (a), we can assume without loss of generality that each MSDL ND in N is also SDL ND, i.e., ND = ND D s.t. SD = SD D.

[Problem 2] Find a condition for some MSDLs, NDi = (SDi, TDi, FDi, MDi), i = 1, 2,..., m, s.t. SDi SDi SDi DM NDm D s.t. SDi S T to be TL under M D, where MSTR NT = (ST, NT, FT, MT) s.t. SDi S T is contained by each MSDL NDi, i = 1, 2,..., m, and M D is the initial marking on S D = D SD. Note that Remarks 3-2 is not applied to this problem.

4 Answer to Problem 1

In Section 4, let us prove the next theorem which is the answer to Problem 1.

[Theorem 1] In the following, note that “conditionally” means that the conditions of Lemma 4-9 are fulfilled for a bounded MSDL ND containing at least one MSTR NT s.t. SD SD SD ST. (1) The converse of Lemma 3-2, i.e., A(v) A(iv) A(iii), is conditionally true for MSDL ND s.t. S SD SD ST. (2) MSDL ND (s.t. S SD SD ST) is TL (B or B) under M D without using any behavioral traps [7] iff at least one MSTR NT (s.t. SD SD ST) is conditionally TL (B or B) under M D.

4.1 Definitions and Properties for ND s.t. SD SD ST

In this subsection, we will show some definitions and some properties for MSDLs with real d-t structure in order to prove Theorem 1. See also Ref.[6].
4.1.1 Some Definitions for $N_D$ s.t. $S_D \supset S_T$

[Definition 4-1] $(C_T, AC_T)$, [6]. Let MSTR $N_T$ (s.t. $S_D \supset S_T$) in $N$ be T-cornucopia iff at least one MSTR $N_T$ s.t. $S_D \supset S_T$ is TL\B under $M_D$ or MSTR $N_T$ s.t. $S_D = S_T$ is TL\B under $M_D$. Let T-cornucopia be denoted $CT$. MSTR $N_T$ s.t. $S_D \supset S_T$ is an absolute T-cornucopia iff MSTR $N_T$ s.t. $S_D \supset S_T$, which is $CT$, can have (i) any number of tokens or (ii) the constant number of tokens to feed any number of tokens to the net of SDI and is CT under $MSTR_{SD2}$ of $\Sigma_D$ = $t_13$. $\Sigma_D = \Sigma_D$, ST, TD, FD, MoD. $\Sigma_D = \Sigma_D$, ST, TD, FD, MoD.

[Example 4-1] Figure 1 (a) is MSDL $N_D$ s.t. $S_D = S_{T1} \cup S_{T2} \cup \{p_5, p_6\}$, where $S_{T1} = \{p_5, p_6, p_7\}$ and $S_{T2} = \{p_1, p_2, p_3, p_4, p_8\}$. Let $\Sigma_D = \Sigma_D$, ST, TD, FD, MoD where $\Sigma_D = \Sigma_D$, ST, TD, FD, MoD.

[Definition 4-2] $(N_{DC}, T^*_D, a_i, k_i, T_o, and \hat{T}_o)$, [6]. First, for $N$, we consider MSDL $N_D = (S_D, T_D, F_D, Mo_D)$, where at least one target MSTR $N_T = (S_T, T_T, F_T, Mo_T)$ is contained in $N_D$ s.t. $S_D \supset \not\subseteq S_T$, $T_D \not\supset \not\subseteq T_T$, and $F_D \not\supset \not\subset F_T$. Secondly, let a directed circuit which contains all the places of $S_D$ be denoted $N_{DC} = (S_D, T_D, F_D, Mo_D)$, where $S_D = S_D$ and $T_D$ is the set of transitions on $N_{DC}$. When $T^*_D \not\subseteq \not\subseteq T_T \not\subseteq \not\subseteq (S_D \supset \not\subseteq S_T)$ is given, $a_i \not\subseteq \not\subseteq t_i \not\subseteq \not\subseteq \not\subset S_T$ and $k_i \not\subseteq \not\subseteq k_i \not\subseteq \not\subseteq \not\subseteq S_T$ for $t_i \in T^*_D$. Let $T_o$ and $\hat{T}_o$ be $T_o \not\subseteq \not\subseteq ((S^*_D \supset \not\subseteq S_T) \not\subseteq \not\subseteq T^*_D \not\subseteq \not\subseteq T_T \not\subseteq \not\subseteq (S_D \supset \not\subseteq S_T) \not\subseteq \not\subseteq T^*_D \not\subseteq \not\subseteq T_T \not\subseteq \not\subseteq S_T$.

[Definition 4-3] $(RP(t_i)$ for $t_i \in T^*_D), [6]$. Consider the firing sequence in connection with liveness (i.e., the worst firing sequence) for MSDL $N_D$ s.t. $S_D \supset S_T$ in $N$ in the following.

(1) Put one token without duplication on some places such those $p \in t^*_i \cap S_T \forall t_i \in T_o$ of all the other MSTRs, except for the target MSTR, in $N_D$ s.t. $S_D \supset \not\subseteq S_T$ under the behavior or firing sequence in connection with liveness, in which let $t_i$ be the total number of the above initial tokens.

(2) If $y_i$ tokens are returned to $(\not\subseteq t_i \cap S_T)$ through the target MSTR $N_T$ and the net of $S_D \supset \not\subseteq S_T$ when $t_i \in T_o$ is fired $\epsilon_i$ times using $\gamma_i$ initial tokens in $N_T$, let us say that $t_i \in T_o$ has $y_i$ return paths, where $t_i \in T_o$ of the other MSTRs, except for the target MSTR, may be fired $\epsilon_i$ times to advance the above process and $\gamma_i$ is the minimum number of required initial tokens on the target MSTR $N_T$ to fire $t_i \in T_o \epsilon_i$ times. For $t_i \in T^*_D \not\subseteq \not\subseteq T_T$, $y_i$ return paths are calculated under the condition that no $t_k \in T_o$ of the target MSTR $N_T$ is fired.

Fig. 1 An example for Lemma 4-6 and Lemma 4-7.

(a) The original net $N$, where MSDL $= SDL$: $S_D = \{p_1, p_2, \ldots, p_8\}$, STR: $S_T = \{p_1, p_2, \ldots, p_8\}$, MSTR$_1$: $S_{T1} = \{p_5, p_6, p_7\}$, and MSTR$_2$: $S_{T2} = \{p_1, p_2, p_3, p_4, p_5, p_8\}$.

(b) The net obtained by reducing only MSTR$_1$ from $N$, where $p^* = \{p_5, p_6, p_7\}$.

(c) The complete reduced net $N'$. 

In Fig. 1, MSTR of $8T_1$ and MSTR of $8T_2$ are examples for $CT \cap AC_T$ under $M_D$. Figure 2 is SDL $N_D$ s.t. $S_D = S_{D1} \cup \{p_5, p_6, p_7\}$, where $S_{D1} = \{p_5, p_6, p_7\}$, $S_{D2} = S_T \cup \{p_5, p_6, p_7\}$, and $S_{D1} \cap S_{D2} = S_T$. In Fig. 2, MSTR of $S_T$ is $CT \cap AC_T$ under $M_D$ against MSDL of $S_{D1}$ and is $CT \cap AC_T$ under $M_D$ against MSDL against MSDL$_2$ (consider the net without $t_13$) of $S_D$. But, note that MSTR of $S_T$ is not $CT$ under $M_D$ against MSDL$_2$ (consider the net with $t_13$) of $S_D$ because of the transition $t_13$. 

In (a), $MSTR_1$ of $S_{T1}$ and $MSTR_2$ of $S_{T2}$ is contained in $S_D \supset S_T$ irrespective of the net structure and marking of $S_D \supset S_T$, provided that no transition $t \in S_T \not\subseteq S_T$ is fired. Let an absolute T-cornucopia be denoted $CT$, where $CT$ can have (i) any number of tokens or (ii) the constant number of tokens to feed any number of tokens to the net of $SD \supset S_T$ irrespective of the net structure and marking of $SD \supset S_T$, provided that no transition $t \in S_T \not\subseteq S_T$ is fired. 

Fig. 2 is SDL $N_D$ s.t. $S_D = \{p_1, p_2, \ldots, p_7\}$, $S_{D1} = \{p_5, p_6, p_7\}$, and $S_{D1} \cup S_{D2} = S_D$. In Fig. 2, MSTR of $S_T$ is $CT \cap AC_T$ under $M_D$ against MSDL of $S_{D1}$ and is $CT \cap AC_T$ under $M_D$ against MSDL against MSDL$_2$ (consider the net without $t_13$) of $S_D$. But, note that MSTR of $S_T$ is not $CT$ under $M_D$ against MSDL$_2$ (consider the net with $t_13$) of $S_D$ because of the transition $t_13$. 

In (b), $MSTR_1$ of $S_{T1}$ and $MSTR_2$ of $S_{T2}$ is contained in $S_D \supset S_T$ irrespective of the net structure and marking of $S_D \supset S_T$, provided that no transition $t \in S_T \not\subseteq S_T$ is fired. Let an absolute T-cornucopia be denoted $CT$, where $CT$ can have (i) any number of tokens or (ii) the constant number of tokens to feed any number of tokens to the net of $SD \supset S_T$ irrespective of the net structure and marking of $SD \supset S_T$, provided that no transition $t \in S_T \not\subseteq S_T$ is fired. Let an absolute T-cornucopia be denoted $CT$, where $CT$ can have (i) any number of tokens or (ii) the constant number of tokens to feed any number of tokens to the net of $SD \supset S_T$ irrespective of the net structure and marking of $SD \supset S_T$, provided that no transition $t \in S_T \not\subseteq S_T$ is fired.
(3) Let us denote the set of return paths for \( t_i \in T'_D \) and its cardinality by \( \text{RP}(t_i) \) and \( |\text{RP}(t_i)| \), respectively.

[Remarks 4-1] (1) Although Definition 4-3 contains some descriptions of the partial marking of tokens, \( \text{RP}(t_i) \) for \( t_i \in T'_D \) is a kind of description of the net structure of \( N_D \) s.t. \( S_D \supseteq S_T \). The details for these are also discussed in Ref.[6].

(2) \( \text{RP}(t_i) \) for \( t_i \in T'_D \) of MSDL \( N_D \) s.t. \( S_D = S_T \) in \( N \) is defined if we take the last part of Definition 4-2 into consideration in Definition 4-3.

(3) For \( t_i \in T'_D \) of the target MSTR \( N_T \) of MSDL \( N_D \) s.t. \( S_D \supseteq S_T \) in \( N \), \( a_i, k_i \leq k_i \) and \( \gamma_i \geq y_i \) \( \Rightarrow \) \( |\text{RP}(t_i)| \). Note that there exists in general no one to one correspondence between \( a_i \leq k_i \) and \( \gamma_i \geq y_i \) in \( N \).

(4) When we find \( N_{DC} \) for MSDL \( N_D \) s.t. \( S_D \supseteq S_T \), we include each transition which is selected in the firing sequence of Definition 4-3.

[Example 4-2] Consider Fig.3 where MSDL of \( N_D = \{ p_1, p_2, \cdots, p_{14} \} \) contains two MSTRs: \( S_{T1} = \{ p_1, p_2, p_3, p_4 \} \) and \( S_{T2} = \{ p_4, p_{10}, p_{11}, p_{12} \} \). For MSTR1, of \( S_{T1}, T_T = T'_T \cup \{ t_1 \}, T'_T = \{ t_1, t_5, t_7 \}, T_o = \{ t_7 \} \) and for MSTR2 of \( S_{T2}, T_T = T'_T \cup \{ t_1 \}, T'_T = \{ t_10, t_{11} \}, T_o = \{ t_{11} \} \). First, consider Fig.3 in connection with Definition 4-2. All the transitions except \( t_5, t_7, t_10, t_{11} \) can be chosen as \( T_{DC} \) of Fig.3. For example, if we choose MSTR1 as the target MSTR \( N_T \), we have \( T_o = \{ t_1, t_{10} \} \). Then, \( a_1 = 2 \) and \( k_1 = 1 \) for \( t_1 \), \( a_4 = 1 \) for \( t_4 \), \( a_7 = 2 \) and \( k_7 = 1 \) for \( t_7 \), and so on. Secondly, consider again Fig.3 in connection with Definition 4-3. If we choose MSTR1 as the target MSTR \( N_T \), then we put one initial token on \( p_{11} \) and \( \beta_i = 1 \). Under this condition, we put two tokens on \( \{ p_1, p_4 \} \) in order to find \( y_i = |\text{RP}(t_i)| \), then we have \( y_1 = 2 = \gamma_1, a_1 = 2, k_1 = 1 \) and \( c_1 = 1 \). In the same way as in \( p_1 \), we have \( y_2 = 1 = \gamma_2, a_2 = e_2 = 1, k_2 = 2 \) for \( t_2 \). \( y_3 = 1 = \gamma_3, a_3 = e_3 = k_3 = 1 \) for \( t_3 \). \( y_i = 3, 4, 5; a_6 = 3, k_6 = e_6 = 1 \) for \( t_6 \). \( y_7 = 2 = \gamma_7, a_7 = 2, k_7 = e_7 = 1 \) for \( t_7 \); and \( y_{12} = 4 = \gamma_{12}, a_{12} = k_{12} = 4, e_{12} = 1 \) for \( t_{12} \). Note that Fig.3 is a special net such that \( \forall t_i \in T'_T : y_i = \gamma_i \), while Fig.1, Fig.2, and Fig.5 are general ones.

Although we use Definitions 4-1 and 4-2 from now on, Definition 4-3 is used in Subsection 4.3.

### 4.1.2 Reduced Nets \( N' \) for \( N \) and Their Properties

First, we will give a basic property for \( N_T \) s.t. \( S_D \nsubseteq S_T \).

[Lemma 4-1] (1) \( \exists \) at least one transition \( t_s \in T_s \) \( \triangleleft (S'_D \cap S'_T) \cap (S_D \setminus S_T)^* \subseteq T_o \) in each \( N_T \) s.t. \( S_D \nsubseteq S_T \).

(2) \( \exists \) at least one transition \( t \in S_T \setminus S'_T \) in each \( N_T \) s.t. \( S_D \nsubseteq S_T \).

(Proof) (1) If there is no transition \( t_s \in T_s \subseteq T_o \), we have \( \forall t_s \in T_s : \phi \subseteq (S'_D \cap S'_T) \cap (S_D \setminus S_T)^* \). Then we have \( \phi \subseteq (S_D \setminus S_T)^* \subseteq (S_D \setminus S_T)^* \) and this means that \( S_D \setminus S_T \) is a new deadlock. However, this contradicts the fact that \( S_D \) was MSDL. Therefore we conclude (1) is true. (2) is due to the net structure of \( N_D \) s.t. \( S_D \nsubseteq S_T \), i.e., the strong connectivity of \( N_D \), and the definition of MSTR \( N_T \) s.t. \( \phi \subseteq S'_T \subseteq S_T \). Q.E.D.

Secondly, we will define the reduced net \( N' \) and give two lemmas concerning \( N_T \) s.t. \( S_D \nsubseteq S_T \).

[Definition 4-4] (Reduction Process for \( N' \)). Let us consider the reduced net \( N' = (S', T', F', M'_o) \) which is obtained from \( N = (S, T, F, M_o) \) by the next procedure; Contract only each \( S_{T_j} \subseteq S_j, j = 1, 2, \cdots, n \), to one place provided that all the transitions and all the arcs are kept unchanged. Let each above new place
Fig.4 Illustration for Lemma 4-3. Note that there exists at least one transition \( t_s \in (p^*_i \cap p_i) \cap (S_D \setminus \{p_i\}) \) which has no arc \((p_k, t_s)\) in each \( p^*_i \subseteq p_i \) in \( N'_D \).

corresponding to \( S_T \) be called the reduced place hereafter.

[Lemma 4-2] For each reduced place \( p_i \in S'_D \), the following three facts are satisfied in \( N'_D = (S'_D, T_D, F'_D, M'_D) \) which is obtained from MSDL \( N_D = (S_D, T_D, F_D, M_D) \) s.t. \( S_D \not\supseteq S_T \) by the reduction process.

1. \( |p_i| \cap |p_i| \geq 1 \).
2. \( |t| \cap |p_i| \geq 1 \).

(Proof) The fact (1) is guaranteed by the definition of each MSTR \( N_T = (S_T, T_T, F_T, M_T) \) s.t. \( \phi \subseteq S'_T \subseteq S_T \) and the reduction process for \( N'_D \). The facts (2) and (3) are guaranteed by the strong connectivity of MSDL \( N_D \) s.t. \( S_D \not\supseteq S_T \) as well as the reasons for (1). Note also that (2) is equivalent to Lemma 4-1 (2). Q.E.D.

[Lemma 4-3] Consider the reduced net \( N'_D \) for MSDL \( N_D \) s.t. \( SD \not\supseteq ST \). For each reduced place \( p_i \in S'_D \), there exists at least one transition \( t_s \in (p^*_i \cap p_i) \cap (S_D \setminus \{p_i\}) \cap (S_D \setminus \{p_i\})^* \), in other words, \( |t_s \cap S'\{p_i\}| = |p_i| = 1 \).

(Proof) Lemma 4-3 is equivalent to Lemma 4-1 (1). Q.E.D.

[Example 4-3] Figure 4 shows illustration for Lemma 4-3, where note that the arc \((p_k, t_s)\) is prohibited.

Consider again Fig.1, where we can find \( t_1 \) as \( t_s \) for \( MSTR_1 \) of \( ST_1 \), and \( t_1 \) and \( t_0 \) as \( t_s \) for \( MSTR_2 \) of \( ST_2 \).

Next, in Subsection 4.2 (4.3, resp.) let us consider Problem for \( N'_D \) s.t. \( S_D \supseteq S_T \): \( TL \cap \emptyset \) under \( M_{oD} \).

4.2 \( N'_D \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \) under \( M_{oD} \)

In this subsection, we will prove that the converse of Lemma 3-2 is always true for \( N'_D \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \) under \( M_{oD} \) of which necessary and sufficient condition is also given such that \( \exists \) at least one MSTR \( N_T \) (s.t. \( S_D \supseteq S_T \); \( TL \cap C_T \) under \( M_{oD} \). However, the necessary and sufficient condition for T-cornucopia (i.e., \( C_T \)) in \( N_D \) (s.t. \( S_D \supseteq S_T \): \( TL \cap \emptyset \) under \( M_{oD} \) is given in Ref. [6].

We have, first, the following two properties (i.e., Lemmas 4-4 and 4-5) for \( N'_D \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \) under \( M_{oD} \).

[Lemma 4-4] (1) \( N'_D \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \) under \( M_{oD} \) --- \( \exists \) at least one MSTR \( N_T \) (s.t. \( S_D \supseteq S_T \): \( TL \cap \emptyset \cap C_T \) under \( M_{oD} \).

2. \( N_T \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \cap C_T \) under \( M_{oD} \). --- Each transition \( t_s \in T_s \subseteq T_0 \) is firable under \( M_{oD} \), where \( T_0 \) is defined in Lemma 4-1 (1).

(Proof) (1) If there exists no MSTR \( N_T \) (s.t. \( S_D \supseteq S_T \)): \( TL \cap \emptyset \cap C_T \) under \( M_{oD} \) in MSDL \( N'_D \) (s.t. \( S_D \supseteq S_T \), each MSTR \( N_T \) (s.t. \( S_D \supseteq S_T \)) is \( TL \cap \emptyset \cap C_T \) under \( M_{oD} \) and then \( N'_D \) (s.t. \( S_D \supseteq S_T \)) becomes \( TL \cap \emptyset \) under \( M_{oD} \) as in Lemma 4-9. However, this contradicts the premise. Therefore (1) is true.

(2) If \( N_T \) is \( AC_T \), this assertion is trivial. If \( N_T \) is \( C_T \cap AC_T \), this property \( C_T \cap AC_T \) of \( N_T \) is guaranteed under the condition that the net of \( S_D \setminus S_T \) satisfies some net structure [6,7]. Therefore each transition \( t_s \in T_s \subseteq T_0 \) is firable under \( M_{oD} \) from the definition of \( C_T \cap AC_T \). Q.E.D.
[Lemma 4-5] \(N_T\) (s.t. \(S_D \not\supseteq S_T\)): \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\). — Each transition \(t \in T_o \Leftrightarrow (S_T^* \cap S_T) \cap \ast(P_T \setminus S_T)\) is firable under \(M_{SD}\).

(Proof) In general, \(T_o\) fulfills also \((S_T^* \cap S_T) \cap (S_D \setminus S_T)^*\). Then, an output transition \(t \in T_o\) from the net of \(S_T\) to the net of \(S_D \setminus S_T\) can be controlled by the net of \(S_D \setminus S_T\) as well as the net of \(S_T\), i.e., the target MSTR \(N_T\). Thus the above output transition \(t \in T_o\) may have two or more input places, but each transition \(t \in T_o\) including each \(t_i \in T_o \subseteq T_o\) is firable under \(M_{SD}\) under the condition of \(N_T\) (s.t. \(S_D \not\supseteq S_T\)): \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\) because of the same reason as that of Lemma 4-4 (2).

Secondly, we will give an interesting property for MSTRs \(N_T\) s.t. \(S_D \not\supseteq S_T\).

[Lemma 4-6] Let us assume that two or more MSTRs are included in MSDL \(N_D\) s.t. \(S_D \not\supseteq S_T\) and an arbitrary pair of MSTR1 \(N_T1 = (S_{T1}, T_{T1}, F_{T1}, M_{TD1})\) and MSTR2 \(N_T2 = (S_{T2}, T_{T2}, F_{T2}, M_{TD2})\) fulfills \(S_{T1} \cap S_{T2} \neq \emptyset\). Then, if one of them is \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\), the other is also \(\text{TL} \cap \text{B} \cap C_T\).

(Proof) Assume that MSTR1 \(N_T1\) is \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\) and MSTR2 \(N_T2\) is unknown to be live or not under \(M_{SD}\). First, reduce only \(N_T1\) in the given MSDL \(N_D\) as in Figs.1 (a) and (b), then MSTR2 \(N_T2\) is modified to have a non-MSTR and non-MSDL subnet because there exists at least one pair of transitions; \(t_i \in (S_{T2} \setminus \{p\}) \setminus (S_{T2} \setminus \{p\})^*\) and \(t_j \in (S_{T2} \setminus \{p\})^* \setminus (S_{T2} \setminus \{p\})^*\) in the modified net, where \(p\) is \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\) and means some places of MSTR2 \(N_{T2}\) as well as all the places of MSTR1 \(N_{T1}\). Secondly, if we take \(N_{TD1}\) and \(N_{TD2}\) under MSDL and Lemma 4-5 into consideration, we can fire repeatedly the above transition \(t_i\). Then MSTR2 \(N_{T2}\) can be made \(\text{TL} \cap \text{B}\) under \(M_{SD}\) because of the next reason: If we encounter, at this time, the situation in which there exist no enabling transitions, this means that there is another MSDL inside the given MSDL \(N_D\) because of ALGORITHM (Deadlock) for finding SDL \(N_D\) s.t. \(\hat{S}_D = S_D\) in Appendix (see also Remarks 3-2 (b)). However, this contradicts the fact that the given \(N_D\) was MSDL. Therefore MSTR2 \(N_{T2}\) is always \(\text{TL} \cap \text{B}\) under \(M_{SD}\) if MSTR1 \(N_{T1}\) is \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\).

Q.E.D.

Finally, we will give a sufficient condition for (1) of Theorem 1, in which \(N_D\) and \(N_T\) s.t. \(S_D \supseteq S_T\) are \(\text{TL} \cap \text{B}\) under \(M_{SD}\).

[Lemma 4-7] If \(\exists\) an MSTR \(N_T\) (s.t. \(S_D \supseteq S_T\)): \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\), (1) of Theorem 1 is unconditionally true, but in this case \(N_D\) (s.t. \(S_D \supseteq S_T\)) is also \(\text{TL} \cap \text{B}\) under \(M_{SD}\).

(Proof) If \(S_D \supseteq S_T\), this is trivial and \(N_D\) (s.t. \(S_D \supseteq S_T\)) is \(\text{TL} \cap \text{B}\) under \(M_{SD}\). Then, consider the case of \(S_D \not\supseteq S_T\) hereafter. Let us have \(N_D\) (s.t. \(S_D \supseteq S_T\)) using [Reduction Process for \(N_D]\). First, each reduced place \(p_i \in S_{Dp}\), i.e., an MSTR, or some MSTRs sharing at least one place in \(N_D\), fulfills the conditions of Lemma 4-3 (see also Lemma 4-6). If \(p_i \in S_{Dp}\) has some tokens at the initial marking, \(t_s\) is always able to fire (whenever necessary, two or more times) and can transfer some tokens to \(p_{i+1} \in S_{Dp}\) remaining at least one token in \(p_i\) because of at least one double arc between \(p_i\) and \(t_s\) (see Fig.4). Secondly, it is true that \(p_{i+1}\) also fulfills the conditions of Lemma 4-3, then some tokens in \(p_{i+1}\) are transferred to \(p_{i+2} \in S_{Dp}\) keeping also at least one token in \(p_{i+1}\) \(\in S_{Dp}\). Repeating this process for all the places of \(p_k \in S_{Dp}\) in \(N_D\) and taking Lemma 4-5 into consideration, we can conclude that \(N_D\), i.e., \(N_D\) (s.t. \(S_D \supseteq S_T\)), is \(\text{TL} \cap \text{B}\) under \(M_{SD}\). Notice that if we encounter the situation in which there exists no enabling transition, remaining some places to cover in \(N_D\) for doing the above process, this means that there is another MSDL in \(N_D\) because of ALGORITHM (Deadlock) for finding SDL s.t. \(\hat{S}_D = S_D\) (see also Remarks 3-2 (b)). However, this contradicts the fact that \(N_D\) was one MSDL.

Q.E.D.

[Example 4-4] In Fig.1 with an initial token on \(p_8\), MSTR1 of \(S_{T1}\) is \(\text{TL} \cap \text{B} \cap AC_T\) under \(M_{SD}\). Then MSTR2 of \(S_{T2}\) is also \(\text{TL} \cap \text{B}\) under \(M_{SD}\) and the whole net \(N_D\) defined by \(S_D = S_{T1} \cup S_{T2} \cup \{p_8,p_9\}\) is \(\text{TL} \cap \text{B}\) under \(M_{SD}\).

Combining Lemma 4-7 and Lemma 4-4 (1), we can give the next property.

[Corollary of Lemma 4-7] MSDL \(N_D\) (s.t. \(S_D \supseteq S_T\)): \(\text{TL} \cap \text{B}\) under \(M_{SD}\) \(\Rightarrow \exists\) at least one MSTR \(N_T\) (s.t. \(S_D \supseteq S_T\)): \(\text{TL} \cap \text{B} \cap C_T\) under \(M_{SD}\).

From this corollary, we can assure that the converse of Lemma 3-2 is always true for \(N_D\) (s.t. \(S_D \supseteq S_T\)): \(\text{TL} \cap \text{B}\) under \(M_{SD}\).

4.3 \(N_D\) (s.t. \(S_D \supseteq S_T\)): \(\text{TL} \cap \text{B}\) under \(M_{SD}\)

In this subsection, we will find the conditions that the converse of Lemma 3-2 is true for the remains of
Problem 1, i.e., $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$, in which we also give the necessary and sufficient condition for $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$.

First, we will give two necessary conditions for $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$, where Definition 4-3 is used.

**[Lemma 4-8]** $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$. — The condition (*) is fulfilled, where the condition (*) means $\forall t_i \in T^]\cap SD \cup ST\mid b_i = \max_j \{b_j \mid t_j \in SD \cup ST\}$. (Proof) If MSDL $ND$ (s.t. $SD \supseteq ST$) is TL\cap B under $MoD$, every MSTR $NT$ (s.t. $SD \supseteq ST$) is also TL\cap B under $MoD$ because of the following two facts: (1) TL under $MoD$ is evident from Lemma 3-2. (2) B under $MoD$ for each MSTR is shown if we consider the contraposition. — That is, if there exists at least one MSTR $NT$ (s.t. $SD \supseteq ST$) which is TL\cap B under $MoD$, we can feed any number of tokens to the net of $SD \cup ST$ because $NT$ is a T-cornucopia. Then this MSDL which contains $NT$: TL\cap B under $MoD$ becomes also TL\cap B under $MoD$. But this contradicts the premise.

Next, we can show that the condition (*) is fulfilled in $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$. If it does not fulfill the condition (*), it becomes B under $MoD$ or dead under $MoD$, because any TL case, except for the above TL cases under the condition (*), is covered by $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$ and each case falls under T-cornucopia [6][7]. Q.E.D.

**[Corollary of Lemma 4-8]** $ND$ (s.t. $SD = ST$): TL\cap B\cap C_T under $MoD$. — The condition (**) is fulfilled, where the condition (**) means $\forall t_i \in T^]\cap SD \cup ST\mid b_i = b_j$, and $t_j \in T^]\cap SD \cup ST\}$, where see notations of Definitions 4-2 and 4-3.

(Proof) This is obtained from Lemma 4-8 if we consider the condition of $SD = ST$ in Lemma 4-8 and take the last part of Definition 4-2 into consideration. Q.E.D.

Secondly, we will give two necessary and sufficient conditions for both $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$ and $ND$ (s.t. $SD = ST$): TL\cap B under $MoD$, where Definition 4-3 is also effectively used.

**[Lemma 4-9]** $ND$ (s.t. $SD \supseteq ST$): TL\cap B under $MoD$. —- $\exists$ at least one MSTR $NT$ (s.t. $SD \supseteq ST$): TL\cap Br\cap C_T under $MoD$, in which $NT$ satisfies the following two conditions, (i) and (ii), under the condition (*) given in Lemma 4-8, where $z = \alpha + \beta_i$, $\alpha$ is the minimum number of initial tokens on $ST$ of the target MSTR $NT$, and $\beta_i$ is the total number of initial tokens in the other MSTRs, except for the target MSTR $NT$, to make this MSDL $ND$ (s.t. $SD \supseteq ST$) live under $MoD$, where see notations of Definitions 4-2 and 4-3.

(i) $\exists \{a_i - k_i\}$ directed paths from $t_i \in T^]\cup SD \cup ST\} \ni \{t_j \cap SD \cup ST\} \ni \{t_j \cap T^]\cup SD \cup ST\} \ni \{t_j \cap T^]\cup SD \cup ST\} \ni \{t_j \cap T^]\cup SD \cup ST\}$, where $\hat{b}_{ij}$ is the maximum number of initial tokens on $t_i \in T^]\cup SD \cup ST\} \ni \{t_j \cap T^]\cup SD \cup ST\}$.

(ii) $\alpha = \max \{\hat{a}_i, \hat{a}_\text{max}\}$, where $\hat{a}_i = \gamma_i$, $\gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\}\ni \{t_j \cap SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\}\ni \{t_j \cap SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\\ni \{t_j \cap SD \cup ST\}\ni \{t_j \cap SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\\ni \{t_j \cap SD \cup ST\}\ni \{t_j \cap SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\\ni \{t_j \cap SD \cup ST\}\ni \{t_j \cap SD \cup ST\}, \gamma_i \ni \gamma_i \Rightarrow \{b_{ij} + k_i \mid t_i \in T^]\cup SD \cup ST\} \ni \{t_j \cap SD \cup ST\}\ni \{t_j \cap SD \cup ST\}... (1)

$\gamma_i = \gamma_i$ for $t_i \in T^]\cup SD \cup ST\}$ with $\hat{b}_{ij}, \hat{a}_\text{max} = \max \{a_i \mid \gamma_i = \gamma_i, t_i \in T^]\cup SD \cup ST\}, k_i$ is the number of return paths due to $k_i$ tokens on $t_i \cap SD \cup ST$ for $t_i \in T^]\cup SD \cup ST\}$ with $\hat{b}_{ij}$.

(Proof) The necessity: First, if $ND$ (s.t. $SD \supseteq ST$) is TL\cap B under $MoD$, we have the condition (*) from Lemma 4-8, where every MSTR $NT$ (s.t. $SD \supseteq ST$) is also TL\cap Br\cap C_T under $MoD$. Secondly, targeting at least one MSTR $NT$ (s.t. $SD \supseteq ST$), in order to guarantee TL under $MoD$, for every MSTR $NT$ under the condition (*), we need at least the minimum number of initial tokens $z = \alpha + \beta_i$. The sufficiency: First, it is evident that, if the target MSTR $NT$ (s.t. $SD \supseteq ST$) satisfies the conditions of Lemma 4-9, $NT$ is TL\cap Br\cap C_T under $MoD$. Secondly, we can show that if the condition (*) is fulfilled in $ND$ (s.t. $SD \supseteq ST$), $ND$ is B under $MoD$ because the condition (*) means that the total number of tokens on $SD$ is bounded under $MoD$. Therefore every MSTR including the target MSTR is also bounded under $MoD$ under the condition (*). Thirdly, we show TL\cap B under $MoD$ for this MSDL $ND$ s.t. $SD \supseteq ST$ if there exists the target MSTR $NT$ (s.t. $SD \supseteq ST$): TL\cap Br\cap C_T under $MoD$ which satisfies the condition (*), (i), and (ii), and $z$ of Lemma 4-9. The condition (*) means $a_i > k_i \forall t_i \in T^]\cup SD \cup ST\}$, because of $\{SD \cup ST\} = \phi$ and $a_i = \gamma_i \forall t_i \in T^]\cup SD \cup ST\} \ni \{t_j \cap SD \cup ST\} \ni \{t_j \cap SD \cup ST\}$ (see Remarks 4-1 (3)). Moreover, $\gamma_i = \gamma_i \forall t_i \in T^]\cup SD \cup ST\}$ should hold because of the condition (*) and then it means that the upper for token count is upper for $\forall t_i \in T^]\cup SD \cup ST\}$; $a_i > k_i$. Now, assume that this MSDL $ND$ is dead under $MoD$ on the above premise. Then there exists at least one dead transition $t_i \in T^]\cup SD \cup ST\}$ under $MoD$ with respect to the target MSTR $NT$. This means also that there exists at least one token-free place $p \in \{t_i \mid p \in M^p \in R(MO)\} \ni \{t_i \mid p \in R(MO)\}$: on the other hand, we defined each return path $RP(t_i)$ for each $t_i \in T^]\cup SD \cup ST\}$ of the target MSTR $NT$ (note that $\gamma_i = [RP(t_i)]$) under the firing sequence for $ND$ as in Definition 4-3, where
If MSDL $N_D$ has at least one MSTR $N_T$ s.t. $S_D \supseteq S_T$: TLnB under $M_{SD}$, the converse of Lemma 3-2 is conditionally true under the conditions of Lemma 4-9. See also Example 4-5.

[Corollary of Lemma 4-9] $N_D$ (s.t. $S_D = S_T$): TLnB under $M_{SD}$. $\iff$ $N_T$ (s.t. $S_D = S_T$): TLnB\wedge AC_T under $M_{SD}$ satisfies $z = \alpha = \bar{a}_{\text{max}} \geq \max_i \{a_i; y_i = \gamma_i = a_i, t_i \in T_T\}$ under the condition (**), i.e., $v_{t_k} \in T_T$: $y_k = \gamma_k = a_k$.

(Proof) This is obvious from Lemma 4-9 if we consider the condition of Lemma 4-9.

Note that the converse of Lemma 4-9 is unconditionally true because $S_D = S_T$.

[Example 4-5] Consider again Fig.3 as an example for Lemma 4-9.

(A) Fig.3 without $t_{21}$: Let us focus on MSTR\textsubscript{1} of $S_T = \{p_1, p_2, p_3, p_4\}$, where $\bar{a}_{\text{max}} = a_1 = 2$ for $t_1$ and $\bar{b}_{ij} = b_{23} = 2 = b_{20} = 3$ for $t_2$ and $t_{20}$. Then we aim $t_4 \in T_T \subseteq T_T$, where $y_6 = \gamma_6 = a_3 = 3, \varepsilon_6 = 1, k_6 = k_6 = 1$ and $\beta_6 = 1$ on $P_{11}$. We have $\bar{b}_{ij} + k' = 3 + 1 = 4 > 3 = \gamma_6$ because of eq.(1a) and $\alpha = \max(\bar{a}, \bar{a}_{\text{max}}) = \bar{a} = 4$ because $\bar{a} = \bar{b}_{ij} + k' = 4$. Therefore we have $z = \alpha + \beta_6 = 4 + 1 = 5$ and this net is t-live and bounded under the above initial marking.

Note that if $z = \alpha + \beta_6 = \gamma_6 + \beta_6 = 3 + 1 = 4$, this net is p-live, but not t-live because of $z t_{20}$. Under $z = 4$, Fig.3 has at least one TL MSTR (i.e., MSTR\textsubscript{2} defined by $S_T = \{p_9, \ldots, p_{12}\}$), but this net is not TL, i.e., is not unconditionally TL. We need the conditions of Lemma 4-9 to guarantee TL for this net as in the above (s.t. $z = 5$).

(B) Fig.3 with $t_{21}$: Let us focus on MSTR\textsubscript{1} of $S_T = \{p_1, p_2, p_3, p_4\}$, where $\bar{a}_{\text{max}} = a_2 = 4$ for $t_2$ and the other quantities remain unchanged. Then we have $\alpha = \max(\bar{a}, \bar{a}_{\text{max}}) = \bar{a} = \bar{a}_{\text{max}} = 4$, where $\bar{a} = \bar{b}_{ij} + k' = 3 + 1 = 4$, and $z = \alpha + \beta_6 = 4 + 1 = 5$. Therefore this net is t-live as well as p-live and bounded under the above initial marking.

If we also choose $z = 3 + 1 = 4$ in this example, we can say that this net is p-live, but not t-live because of $z t_{20}$ and $t_{21}$. Then this example has at least one TL MSTR (i.e., MSTR\textsubscript{2}), but this net is not unconditionally TL. We also need the conditions of Lemma 4-9 to guarantee TL for this example as in the above (s.t. $z = 5$).

Note that it is also easy to give examples for eq.(1b). See Ref.[6].

[Remarks 4-2] (1) MSDL $N_D$ s.t. $S_D \supset S_T$ is PLnB under $M_{SD}$ iff the condition (*) in Lemma 4-8 is fulfilled.

(2) MSDL $N_D$ s.t. $S_D = S_T$ is PLnB under $M_{SD}$ iff the condition (**) in Corollary of Lemma 4-8 is fulfilled.

(3) The converse of Lemma 3-2, i.e., (1) of Theorem 1, is conditionally true when at least one MSTR $N_T$ (s.t. $S_D \supset S_T$): TLnB under $M_{SD}$ satisfies the conditions of Lemma 4-9, where MSDL $N_D$ is also TLnB under $M_{SD}$, while it is unconditionally true when (a) or (b) is satisfied: (a) MSTR $N_T$ (s.t. $S_D = S_T$) is TLnB\wedge AC_T under $M_{SD}$ as in Corollary of Lemma 4-9 and (b) at least one MSTR $N_T$ (s.t. $S_D \supset S_T$) is TLnB\wedge CT under $M_{SD}$ as in Lemma 4-9. Note that the above three cases cover all real d-t properties, but are only sufficient conditions for A(iii) because of Remarks 3-1 (7).

(4) Note that T-cornucopias in $N_D$ (s.t. $S_D \supset S_T$): TLnB under $M_{SD}$ have been clarified by Lemma 4-9 and its corollary (cf. Lemma 4-7 and its corollary).

4.4 Proof for Theorem 1

From the results in Subsections 4.2 and 4.3, we can prove Theorem 1.

(1) of Theorem 1 is true from Lemma 4-7 for $N_T$ (s.t. $S_D \supset S_T$): TLnB\wedge CT under $M_{SD}$, and from Lemma 4-9 and its Corollary for $N_T$ (s.t. $S_D \supset S_T$): TLnB (CT for $S_D \supset S_T$ and AC_T for $S_D = S_T$) under $M_{SD}$. Note that “conditionally” in Theorem 1 means that the conditions of Lemma 4-9 are satisfied for a bounded MSDL $N_D$ containing at least one MSTR $N_T$ s.t. $S_D \supset S_T$ (see also Remarks 4-2 (3)). Let us prove (2) of Theorem 1 next. The necessity for (2) is due to Lemma 3-2 under the condition that no behavioral traps are used. The sufficiency for (2) is due to (1) of Theorem 1, where both Lemma 4-7 and Lemma 4-9 as well as its
Corollary are the conditions due to at least one MSTR $N_T$ s.t. $S_D \supseteq S_T$ in the sense of "conditionally". Q.E.D.

Therefore if we want to guarantee local liveness for each MSDL $N_D$ s.t. $S_D \supseteq S_T$ in general Petri nets $N$, it is enough to choose one MSTR $N_T$ as the target MSTR for each $N_D$ s.t. $S_D \supseteq S_T$ in $N$. However, note that this is a part of local liveness conditions for $N_D$ as defined in A(iii) in Subsection 3.1 because of Remarks 3-1 (6) and (7).

Theorem 1 is useful for expanding the net classes which have nice and useful initial-marking-based liveness conditions, for synthesizing live Petri nets [4-7], and for obtaining liveness condition for general Petri nets, which is the final goal. The necessary and sufficient conditions for $T$-cornucopias in $N_D$ (s.t. $S_D \supseteq S_T$): $T \land B$ under $M_{aD}$ are given in Ref.[6]. See also Remarks 4-2 (4).

5 Answer to Problem 2

We consider a simple answer to Problem 2 in this section, but this is only a sufficient condition and the related complete descriptions for Problem 2 are discussed in Refs.[4,5].

[Theorem 2] When some MSDLs, $N_D_i = (S_{Di}, T_{Di}, F_{Di}, M_{aDi})$, $i = 1, 2, \ldots, m$, fulfill $S_{Di} \cap S_{Dj} \cap \cdots \cap S_{D_m} \neq \emptyset$ and contain at least one MSTR $N_T = (S_T, T_T, F_T, M_{aT})$ in common, each $N_D_i$ (s.t. $S_{Di} \supseteq S_T$) is $T \land B$ under $M_{aD}$ if $N_T$ is $T \land B \land C_T$ under $M_{aD}$ against each $N_D_i$ ($i = 1, 2, \ldots, m$), where $\tilde{N}_D = (\tilde{S}_D, \tilde{T}_D, \tilde{F}_D, M_{aD})$ is defined by $\tilde{S}_D = \sum_{i=1}^{m} S_{Di}$.

(Proof) First, the next is evident; If at least one MSTR $N_T$ in common among MSDLs ($N_D_i, i = 1, 2, \ldots, m$) is $T \land B \land C_T$ under $M_{aD}$, each $N_D_i$ is also $T \land B$ under $M_{aD}$ by Lemma 4-7. Secondly, we want to prove that each $N_D_i$ (s.t. $S_{Di} \supseteq S_T$) is $T \land B$ under $M_{aD}$ even if we consider global liveness under $M_{aD}$ for each $N_D_i$ in $\tilde{N}_D$. Suppose that we have $\tilde{N}_D'$ from $\tilde{N}_D$. In this case, it is true in $\tilde{N}_D'$ that there are some transitions without unique firing condition because of two or more input places in both MSDLs ($N_D_i$) and MSDLk ($N_D_k$), but if we take the fact of $N_T$: $T \land B \land C_T$ under $M_{aD}$ to each $N_D_i$ ($i = 1, 2, \ldots, m$) into consideration, each $N_D_i$ ($i = 1, 2, \ldots, m$) becomes $T \land B$ under $M_{aD}$. Q.E.D.

In Theorem 2, note that we assumed the rather strong condition concerning structure and the initial marking such that MSTR $N_T$ is $T \land B \land C_T$ under $M_{aD}$. See Ref.[4, 5] for the general discussion on Problem 2.

[Example 5-1] Let us show a simple example which does not satisfy the conditions of Theorem 2. In Fig.2, $S_T = \{p_1, p_2, \ldots, p_8\}$, $S_{D1} = S_T \cup \{p_8, p_7\}$, $S_{D2} = S_T \cup \{p_8, p_9, p_10\}$, $S_{Di} \cap S_{Dj} \supseteq S_T$, and $\tilde{S}_D = S_{D1} \cup S_{D2}$-MSTR $N_T$ defined by $\tilde{S}_D$ is $T \land B \land C_T$ under $M_{aD}$ against MSDL $N_{D1}$ defined by $S_{D1}$ and is $T \land B \land C_T$ under $M_{aD}$ against MSDL $N_{D2}$ defined by $S_{D2}$. The whole net $\tilde{N}_D$ is dead under $M_{aD}$ because of the transition $t_{13}$, where note that $\tilde{N}_D$: $T \land B$ under $M_{aD}$ and $N_{D2}$: $T \land P \land B$ under $M_{aD}$. If we insert the arc $(t_{13}, p_1)$ into Fig.2 or insert the arc $(t_{13}, p_9)$ and reverse the direction of (p_10, t_{13}) in Fig.2, then the new net satisfies the conditions of Theorem 2 and $N_{D1}$: $T \land B$ under $M_{aD}$ ($i = 1, 2$).
Next, let us show another rather complex example which satisfies the conditions of Theorem 2. In Fig. 5 (a), the whole net \( N \) (the subnets, resp.) with respect to \( \{P_1, P_2, \ldots, P_{23}\} \) \( \{P_1, P_2, \ldots, P_{19}\} \) \( \{P_1, P_2, \ldots, P_{15}\} \) \( \{P_1, P_2, P_3, P_4\} \) \( \{P_5, P_6\} \) \( \{P_7, P_8, P_9\} \) \( \{P_{16}, P_{17}, P_{18}\} \) \( \{P_{19}, P_{20}, P_{21}\} \) is SDL \( \text{NSD} \) (MSDL \( 1 \) \( \text{NSD} \) \( 1 \), MSTR \( 1 \) \( \text{NT} = \{\text{ST}, \text{TT}, \text{FT}, \text{MT}\} \) to be t-live (bounded or unbounded) under \( \text{MN} \). Figure 5 (b) is the reduced net \( \text{NSD} \) obtained from \( \text{NSD} \) by the reduction process, where \( P_1, P_5, P_7, P_{16}, \) and \( P_{19} \) means \( \text{NT}, \text{NT}_2, \text{NT}_3, \text{NT}_4, \) and \( \text{NT}_5 \), respectively and they are reduced places.

In Fig. 5, \( p_5 \) of \( \text{NSD} \), i.e., \( \text{NT}_2 \) of \( \text{NSD} \), is one of the essential MSTRs (i.e., \( \text{NT}_j, j = 1, 2, 3 \)) to make the net \( \text{NSD} \) live under \( \text{MN} \), where \( \text{MN} \) has at least one token in \( p_5 \) or \( p_6 \) in Fig. 5 (a).

6 Conclusion

As one of useful necessary liveness conditions for general Petri nets, we have shown, in Theorem 1 under the condition that no behavioral traps are used, that the necessary and sufficient condition for local liveness of a minimal deadlock (MSDL; \( \text{NSD} = (\text{SD}, \text{TD}, \text{FD}, \text{MN}) \) s.t. \( \text{SD} \supseteq \text{ST} \), i.e., \( \text{NSD} \) with real d-t structure) is "conditionally" at least one target minimal structural trap (MSTR; \( \text{NT} = (\text{ST}, \text{TT}, \text{FT}, \text{MT}) \) to be t-live (bounded or unbounded) under \( \text{MN} \), where "conditionally" means that the conditions of Lemma 4-9 are satisfied for a bounded MSDL \( \text{NSD} \) containing at least one MSTR \( \text{NT} \) s.t. \( \text{SD} \supseteq \text{ST} \) (see also Remarks 4-2 (3)). Therefore based on the above results we can choose one MSTR \( \text{NT} \) s.t. \( \text{SD} \supseteq \text{ST} \) as the target MSTR to locally activate this MSDL \( \text{NSD} \). In order to prove this fact, we have introduced some new concepts such as T-cornucopias and return paths into real deadlock-trap structure (i.e., \( \text{SD} \supseteq \text{ST} \)) in general Petri nets and have also presented the necessary and sufficient condition for MSDL \( \text{NSD} \) s.t. \( \text{SD} \supseteq \text{ST} \) to be t-live and bounded under \( \text{MN} \) (see Corollary of Lemma 4-7) and t-live and bounded under \( \text{MN} \) (see Lemma 4-9 and its Corollary). However, in Lemma 4-7 and its corollary of this paper, we have only used the concept for T-cornucopias and then the necessary and sufficient conditions for T-cornucopias in \( \text{NSD} \) (s.t. \( \text{SD} \supseteq \text{ST} \)) to be explored elsewhere (see Refs.[6,7]). Moreover, we have shown a simple sufficient condition for Problem 2 as in Theorem 2.

The results obtained in this paper are very important for expanding the net classes which have useful and initial-marking-based necessary and sufficient liveness conditions, for synthesizing live (bounded or unbounded) Petri nets, and for obtaining the necessary and sufficient liveness condition [4-7], which is the final goal, for general Petri nets.

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References


Appendix: ALGORITHM (Deadlock)

If a general Petri net $N = (S, T, F, M_0)$ is dead under $M_0$, there exists at least one dead transition $t_i \in T$ under an arbitrary marking $M' \in R(M)$, $M \in R(M_0)$, where at least one place $p_i \in *t_i$ is token-free under $M' \in R(M)$ or all the places of $p_i \in *t_i$ cannot simultaneously be marked under $M' \in R(M)$. If we search all transitions $*p_i$, all places $*p_i$, and so on, we can find at least one dead deadlock with respect to $t_i \in T$.

Then we have the next algorithm.

ALGORITHM (Deadlock)

$S$: the set of given places. $T$: the set of given transitions. $S_5$: the set of places in deadlocks with respect to $t_i \in T$. $T_5$: the set of transitions in deadlocks with respect to $t_i \in T$.

< 1 > Let $k = 1$, $S_5 = \phi$, and $T_5 = \phi$; $t_k = t_1$ is the given dead transition.

< 2 > If $S = \phi$, then go to < 8 >. Or else, then go to the next step < 3 >.

< 3 > Let $*t_k = \{ P_{k+1}^{(1)}, P_{k+1}^{(2)}, \ldots, P_{k+1}^{(m_i)} \}$ be the set of unsearched input places with respect to $t_k$.

If $*t_k = \phi$, then go to < 4 >. Or else, $P_{k+1} = P_{k+1}^{(1)}$, then go to < 5 >.

< 4 > Let $T(S(S))$ be the set of seached places and let $T(S(T))$ be the set of seached transitions.

Find $t_i \in T$ backwardly from $t_k$ s.t. $*t_i \cap (S(S)) \neq \phi$, then let $S = S \{ \{ \text{places on the path from } t_k \text{ to } t_i \} \}$ and $T = T \{ \{ \text{transitions on the path from } t_k \text{ to } t_i \} \}$. Renew the numbering for $S$ and $T$. Let $t_{k+1} = t_i$ and then go to < 7 >.

Or else, let $S = S \{ \{ \text{places on the path from } t_k \text{ to } t_i \} \}$. Renew the numbering for $S$ and $T$. Let $t_{k+1} = t_i$ and then go to < 7 >.

< 5 > Let $P_{k+1} = \{ t_{k+1}^{(1)}, t_{k+1}^{(2)}, \ldots, t_{k+1}^{(m_i)} \}$ be the set of unsearched input transitions with respect to $P_{k+1}$.

If $P_{k+1} = \phi$, then go to < 6 >. Or else, $t_{k+1} = t_{k+1}^{(1)}$, then go to < 7 >.

< 6 > Find $p_i \in S$ backwardly from $p_{k+1}$ s.t. $*p_i \cap (T(T)) \neq \phi$, then $S_5 = \{ \{ \text{places on the path from } p_{k+1} \text{ to } p_i \} \}$ and $T_5 = \{ \{ \text{transitions on the path from } p_{k+1} \text{ to } p_i \} \}$. Let $S = S \{ S_5 \} \text{ and } T = T \{ T_5 \}$. Renew the numbering for $S$ and $T$. Let $p_{k+1} = p_i$, then go to < 5 >.

Or else, let $S_5 = \{ \{ \text{places on the path from } p_{k+1} \text{ to } t_i \} \}$. Renew the numbering for $S$ and $T$. Let $t_{k+1} = t_i$ and then go to < 7 >.

< 7 > $k = k + 1$ and go to < 2 >.

< 8 > Output $S_5$ and $T_5$, then end.

Through ALGORITHM (Deadlock), we can find all maximal structural deadlocks defined by $S_5$, from each of which there exists at least one directed path to $t_i \in T$. Now, if we input only MSDL $N_D$ s.t. $S_D = S_D$, ALGORITHM (Deadlock), we can verify non-liveness of $N_D$. 

