Uniform Approximations to Finite Hilbert Transform and its Derivative

Takemitsu Hasegawa*
Department of Information Science
Fukui University, Fukui, 910-8507, Japan

Received 28 November 2002
Revised 27 May 2003

Abstract

Interpolatory integration rules of numerical stability are presented for approximating Cauchy principal value (pv) integrals
\[ \int_{-1}^{1} \frac{f(t)}{t - c} \, dt \]
and Hadamard finite-part (fp) integrals
\[ \int_{-1}^{1} \frac{f(t)}{(t - c)^2} \, dt, \quad -1 < c < 1, \]
respectively, for a given smooth function \( f(t) \). Present quadrature rules consist of interpolating \( f(t) \) at abscissae in the interval of integration \([-1, 1]\) except for the pole \( c \), where neither the function value \( f(c) \) nor its derivative \( f'(c) \) is required, followed by subtracting out the singularities.

We demonstrate that the use of both endpoints \( \pm 1 \) as abscissae in interpolating \( f(t) \) is essential for uniformly approximating the integrals, namely for bounding the approximation errors independently of the values of \( c \). In fact, for the fp integrals the use of double abscissae at both endpoints \( \pm 1 \) as well as simple abscissae in \((-1, 1)\) enables the uniform approximations, while the use of simple abscissae at both endpoints \( \pm 1 \) and those in \((-1, 1)\) is sufficient for the pv integrals. These facts suggest that finite Hilbert transforms (pv integrals) and their derivatives (fp integrals) with varied values of \( c \) could be approximated efficiently with the same number of abscissae, respectively. Some numerical examples are given.

Keyword:
quadtrature rules, Cauchy principal value integrals, Hadamard finite-part integrals, error analysis

1 Introduction

We propose uniform approximation methods to finite Hilbert transform (Cauchy principal value integral; p.v.) given by

\[ Q(f; c) := \int_{-1}^{1} \frac{f(x)}{x - c} \, dx, \quad -1 < c < 1, \]  \quad (1.1)

and its derivative (Hadamard finite part integral; f.p.):

\[ Q^H(f; c) := \frac{d}{dc} Q(f; c) = \int_{-1}^{1} \frac{f(x)}{(x - c)^2} \, dx, \]  \quad (1.2)

respectively, where \( f(x) \) are assumed to be smooth functions. Among literature available on quadrature rules for approximating the p.v. and f.p. integrals; see

*email:hasegawa@agauss.fuis.fukui-u.ac.jp
of values of the quadrature rules (1.6) and (1.7) provide efficient methods that could approxi-
mate the finite Hilbert transform (1.1) and its derivative (1.2) for a variety of cases. Both rules
(1.6) with respect to \( c \) and (1.7) are needed for each value of \( c \) within the same accuracies and with the same numbers of function evaluations except that the evaluations of \( f(c) \) and the derivative \( f'(c) \) in (1.6) and (1.7) are needed for each value of \( c \).

The purpose of this paper is to show that without the evaluations of \( f(c) \)'s and \( f'(c) \)'s it is possible to construct the quadrature rules of uniform convergence for the p.v. and f.p. integrals in the same form as (1.6) and (1.7), where \( f(c) \) and \( f'(c) \) are replaced by the interpolating polynomial \( p_N(c) \) (1.5) and \( p'_N(c) \), respectively. In particular, we demonstrate that the endpoints \( \pm 1 \) of \([-1, 1]\) in

\[
Q(f; c) = \int_{-1}^{1} g_c(x) dx + f(c) \log \left( \frac{1-c}{1+c} \right),
\]

(1.3)

where \( g_c(x) \) is defined by

\[
g_c(x) = \frac{f(x) - f(c)}{(x-c)},
\]

(1.4)

followed by applying some ordinary quadrature rule, such as Gaussian or Newton-Cotes rule, to the integral in the right-hand side of (1.3). Although this scheme is simple and efficient in general, we have severe numerical cancellation in evaluating \( g_c(x) \) at a node \( x_j \) of the rule if \( x_j \) happens to be very close to \( c \).

For the efficient computation of the p.v. integrals Hasegawa and Torii [12] propose an automatic quadrature method of numerical stability; the instability above is avoided by approximating the function \( f(x) \) (and \( f'(c) \)) in (1.4) by an interpolating polynomial \( p_N(x) \) (and \( p_N(c) \)) to evaluate the divided difference \( g_c(x) \) without the cancellation. In fact, let \( T_k(x) \) be Chebyshev polynomial of the first kind given by

\[
T_k(x) = \cos(k \theta),
\]

where \( x = \cos \theta \). Then, they write the polynomial \( p_N(x) \) in terms of the finite sum of \( T_k(x) \),

\[
p_N(x) := \sum_{k=0}^{N} a_k^N T_k(x),
\]

(1.5)

where the double prime denotes the summation whose first and last terms are halved. The approximate integral \( Q_N(f; c) \) can be written as follows [13]:

\[
Q_N(f; c) = 4 \sum_{k=0}^{N-1} A_k^N T_k(c) + f(c) \log \left( \frac{1-c}{1+c} \right),
\]

(1.6)

where the prime denotes the summation whose first term is halved and \( A_k^N \) are the coefficients independent of \( c \) and defined by

\[
A_k^N = \frac{1}{2} \sum_{n=0}^{[(N-k-1)/2]} \frac{a_{2n+k+1}^N}{2n+1}, \quad 0 \leq k \leq N-1,
\]

and we take \( a_{N/2}^N \) instead of \( a_N^N \). Further, differentiating the approximation (1.6) with respect to \( c \) yields an approximation to the f.p. integral (1.2) [13]:

\[
Q_N^H(f; c) = 4 \sum_{k=1}^{N-1} k A_k^N U_{k-1}(c) + f'(c) \log \left( \frac{1-c}{1+c} \right) - \frac{2f(c)}{1-c^2}.
\]

(1.7)

Both rules \( Q_N(f; c) \) (1.6) and \( Q_N^H(f; c) \) (1.7) are shown [12, 13] to be not only numerically stable but uniformly convergent; the approximation errors can be bounded uniformly, namely, independently of the pole \( c \). This implies that the quadrature rules (1.6) and (1.7) provide efficient methods that could approximate the finite Hilbert transform (1.1) and its derivative (1.2) for a variety of values of \( c \) within the same accuracies and with the same numbers of function evaluations except that the evaluations of \( f(c) \) and the derivative \( f'(c) \) in (1.6) and (1.7) are needed for each value of \( c \).
The integral in (2.1) is the same as the first term of the right-hand side of (1.6), which is derived from (1.5), (A.2) and the fact that if \( k \) is even, otherwise vanishes. The sample points \( x_j \) used to interpolate \( f(x) \) are \( x_j = \cos(\pi j/N) \), \( j = 0, \ldots, N \), which are zeros of the polynomial [12, 13] defined by

\[ \omega_{N+1}(x) = T_{N+1}(x) - T_{N-1}(x) = 2(x^2 - 1)U_{N-1}(x), \]

(2.2)

where \( U_k(x) \) is the Chebyshev polynomial of the second kind defined by \( U_k(x) = \sin(k + 1)\theta/\sin \theta \), \( x = \cos \theta \). The interpolation condition \( f(\cos \pi j/N) = p_N(\cos \pi j/N) \), \( 0 \leq j \leq N \), determines the coefficients \( a_N^k \) of \( p_N(x) \) (1.5) as follows:

\[ a_N^k = \frac{2}{N} \sum_{j=0}^{N} f(\cos(\pi j/N)) \cos(\pi k j/N), \quad 0 \leq k \leq N. \]

(2.3)

It is known that the right-hand side of (2.3) can be efficiently computed by means of the fast Fourier transform (FFT) [11, 15].

Criscuolo and Scuderi [3] propose an interpolatory product quadrature rule of open type based on the abscissae \( \cos(j - 1/2)\pi/N \), \( j = 1, \ldots, N \). They prove that their quadrature rule is uniformly convergent with some restrictions on the location of \( c \). On the other hand, Our quadrature rule is a closed-type rule since both endpoints \( \pm 1 \) are used in interpolating \( f(x) \), as seen in (2.2). In this section we prove that the error of the interpolating polynomial \( p_N(x) \) (1.5) can be expressed in terms of a contour integral [7, 10], which is also expanded in a Chebyshev series [14]:

\[ f(x) - p_N(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{\omega_{N+1}(x) f(z) dz}{(z-x) \omega_{N+1}(z)} = \omega_{N+1}(x) \sum_{k=0}^{\infty} V_k^N(f) T_k(x), \]

(2.4)

where the coefficients \( V_k^N(f) \) are given by

\[ V_k^N(f) = \frac{1}{\pi^2 i} \oint_{\mathcal{E}_\rho} \frac{\tilde{U}_k(z) f(z) dz}{\omega_{N+1}(z)}, \quad k \geq 0. \]

(2.5)
The Chebyshev function of the second kind, $\widetilde{U}_k(z)$, is defined by

$$\widetilde{U}_k(z) = \int_{-1}^{1} \frac{T_k(x)}{(z - x)\sqrt{1 - x^2}} \, dx = \frac{2\pi}{(u - u^{-1})u^k}.$$  

Using (2.4) in (1.1) with $f$ being replaced by $f - p_N$ yields the error of the approximation $Q(p_N; c)$ (2.1):

$$Q(f; c) - Q(p_N; c) = Q(f - p_N; c) = \sum_{k=0}^{\infty} \Omega_k N(c) V_k N(f), \quad (2.6)$$

where $\Omega_k N(c)$ is given by

$$\Omega_k N(c) = \int_{-1}^{1} \frac{\omega N + 1(x) T_k(x)}{x - c} \, dx, \quad -1 < c < 1. \quad (2.7)$$

Now, we show that $|\Omega_k N(c)|$ is bounded independently of $c$.

**Lemma 2.1** Let $\Omega_k N(c)$ be defined by (2.7). Then $\Omega_k N(c)$ is bounded independently of the value of $c$ as well as $N$ and $k$ as follows:

$$|\Omega_k N(c)| \leq 32/3. \quad (2.8)$$

**Proof:** Let $\Omega_k N(c) = I_k N(c) + J_k N(c)$, where $I_k N(c)$ and $J_k N(c)$ are defined by

$$I_k N(c) = \int_{-1}^{1} \frac{\omega N + 1(x) T_k(x) - \omega N + 1(c) T_k(c)}{x - c} \, dx, \quad (2.9)$$

and

$$J_k N(c) = \omega N + 1(c) T_k(c) \log\{(1 - c)/(1 + c)\}, \quad (2.10)$$

respectively. Hasegawa and Torii [12] proves that $|I_k N(c)| \leq 8$. It remains to prove that $|J_k N(c)| \leq 8/3$. Since $U_{N-1}(c) = \sin N\phi / \sin \phi$ and $T_k(c) = \cos k\phi$ if we set $c = \cos \phi$, we have from (2.2)

$$|\omega N + 1(c) T_k(c)|/2 \leq |(c^2 - 1) U_{N-1}(c)| = |\sin \phi \sin N\phi| \leq \sqrt{1 - c^2}. \quad (2.11)$$

From (2.10) and (2.11) we see that $|J_k N(c)| \leq |h(c)|$, where we define

$$h(x) = 2\sqrt{1 - x^2} \log\{(1 - c)/(1 + c)\}.$$  

It is easy to verify that $|J_k N(c)| \leq 8/3$ because

$$\max_{|x| \leq 1} |h(x)| = 2|h(\pm 0.833 \cdots)| < 2 \times 1.33 < 8/3. \quad \square$$

Finally, from (2.6) and (2.8) we have the following theorem.

**Theorem 2.2** Assume that $f(z)$ is single-valued and analytic inside and on $E_\rho$. Then, the error $Q(f - p_N; c)$ of the approximate integral $Q(p_N; c)$ given by (2.1) is bounded independently of $c$ by

$$|Q(f - p_N; c)| \leq (32/3) \sum_{k=0}^{\infty} |V_k N(f)|, \quad (2.12)$$

where $V_k N(f)$ is given by (2.5).

Suppose that $f(z)$ be a meromorphic function with a finite number of simple poles outside $E_\rho$. Then performing the contour integral of $V_k N(f)$ (2.5) yields $V_k N(f) = O(r^{-k-N})$ when $N \to \infty$, where $r(>1)$ is a constant [12]. From this fact and (2.12) we can see that the error of the quadrature rule (2.1) goes to zero uniformly as $N \to \infty$. 

4
3 Quadrature of f.p. integral and error analysis

Simply differentiating \( Q(p_N; c) \) (2.1) with respect to \( c \) would yield the quadrature rule \( dQ(p_N; c)/dc \) for the f.p. integral in the same form as \( Q^H_Q(f; c) \) (1.7) except that \( f(c) \) and \( f'(c) \) are replaced with \( p_N(c) \) and \( p'_N(c) \). This simple method of derivation, however, could not give an approximation of uniform convergence. In this section, we show that it is required to interpolate \( f(x) \) by using double abscissae at the endpoints \( \pm 1 \) as well as the internal abscissae in \((-1, 1)\), in order to obtain approximations to the f.p. integrals whose error are bounded uniformly, namely independently of \( c \).

Indeed, at the abscissae \( \{\cos(\pi j/N)\}_{j=0}^{N} \cup \{-1, 1\} \), which are the zeros of \((x^2 - 1)\omega_{N+1}(x), \) we interpolate \( f(x) \) by the polynomial \( q_{N+2}(x) \) of degree \( N+2 \), which is written in the Newton form using \( p_N(x) \) and expressed in terms of Chebyshev polynomials again as follows:

\[
q_{N+2}(x) := p_N(x) - \omega_{N+1}(x)\{d_1^N + d_2^N U_1(x)\} =: \sum_{k=0}^{N+2} a_k^{N+2} T_k(x). \tag{3.1}
\]

The new coefficients \( d_1^N \) and \( d_2^N \) are determined so that \( q_{N+2}'(x) \) agrees with \( f'(x) \) at \( x = \pm 1 \).

**Lemma 3.1** Let \( N \) be an even integer and \( q_{N+2}(x) \) be a polynomial defined by (3.1). Then the coefficients \( d_1^N \) and \( d_2^N \) are given by

\[
d_1^N = \frac{1}{4N} \left\{ \sum_{k=0}^{N/2-1} (2k + 1)^2 a_{2k+1}^N - \frac{f'(1) + f'(-1)}{2} \right\}, \tag{3.2}
\]

\[
d_2^N = \frac{1}{8N} \left\{ \sum_{k=0}^{N/2-1} (N - 2k)^2 a_{N-2k}^N + \frac{f'(-1) - f'(1)}{2} \right\}, \tag{3.3}
\]

respectively.

**Proof:** We determine \( d_1^N \) and \( d_2^N \) so that \( f'(x) \) may agree with \( q_{N+2}'(x) \) at \( x = \pm 1 \). From (3.1) we have

\[
q_{N+2}'(\pm 1) = p_N'(\pm 1) - \omega_{N+1}'(\pm 1)\{d_1^N + d_2^N U_1(\pm 1)\}. \tag{3.4}
\]

Since \( T_k'(t) = k U_{k-1}(t) \) and \( U_{k-1}(\pm 1) = k(\pm 1)^{k-1}, \) it follows from (1.5) and (2.2) that

\[
p_N'(\pm 1) = \sum_{k=0}^{N-1} (\pm 1)^{k+1} (N - k)^2 a_{N-k}^N, \quad \omega_{N+1}'(\pm 1) = 4N. \tag{3.5}
\]

Finally, using (3.5) in (3.4) and the condition that \( f'(\pm 1) = q_{N+2}'(\pm 1) \), we have

\[
f'(\pm 1) = \sum_{k=0}^{N-1} (\pm 1)^{k+1} (N - k)^2 a_{N-k}^N - 4N( d_1^N \pm 2d_2^N),
\]

from which \( d_1^N \) and \( d_2^N \) are solved as shown in (3.2) and (3.3). \( \square \)

Approximating \( f(x) \) in \( Q(f; c) \) (1.1) by \( q_{N+2}(x) \) (3.1) and differentiating the resulting \( Q(q_{N+2}; c) \) with respect to \( c \) yield a quadrature rule for the f.p. integral as follows:

\[
Q_{N+2}^H(f; c) := \frac{d}{dc} Q(q_{N+2}; c) = \sum_{k=1}^{N+1} k A_k^{N+2} U_{k-1}(c) - \frac{2 q_{N+2}(c)}{1 - c^2} + \frac{dq_{N+2}(c)}{dc} \log \left( \frac{1 - c}{1 + c} \right). \tag{3.6}
\]
where
\[ A_k^{N+2} := 4 \sum_{n=0}^{[(N-k+1)/2]} \frac{a_{2n+k+1}}{2n+1}, \quad 0 \leq k \leq N+1, \]
\[ \frac{dq_{N+2}(c)}{dc} = \sum_{k=1}^{N+2} k a_k^{N+2} U_{k-1}(c). \]

Similarly to section 2 we estimate the error \( dQ(f - q_{N+2}; c)/dc \) of the approximation \( dQ(q_{N+2}; c)/dc \) (3.6). We begin with the interpolation error \( f(x) - q_{N+2}(x) \), which is expressed in terms of the contour integral and expanded in the Chebyshev series as follows:
\[ f(x) - q_{N+2}(x) = \frac{1}{2\pi i} \int_{C_n} \frac{(x^2 - 1) \omega_{N+1}(x) f(z) dz}{(z - x)(z^2 - 1) \omega_{N+1}(z)} \]
\[ = (x^2 - 1) \omega_{N+1}(x) \sum_{k=0}^{\infty} V_k^{N+2}(f) T_k(x), \quad (3.7) \]
where
\[ V_k^{N+2}(f) := \frac{1}{\pi i} \int_{C_n} \frac{\tilde{U}_k(z) f(z) dz}{(z - x)(z^2 - 1) \omega_{N+1}(z)}. \quad (3.8) \]

Using \( f - q_{N+2} \) (3.7) in \( f \) of (1.2) yields the error of the approximation \( dQ(q_{N+2}; c)/dc \) (3.6) for the f.p. integral:
\[ \frac{d}{dc} Q(f - q_{N+2}; c) = \sum_{k=0}^{\infty} \frac{d}{dc} \Omega_k^{N+2}(c) V_k^{N+2}(f), \quad (3.9) \]
where
\[ \Omega_k^{N+2}(c) := \int_{-1}^{1} \frac{(x^2 - 1) \omega_{N+1}(x) T_k(x)}{x - c} dx, \quad -1 < c < 1. \quad (3.10) \]

Now, we prove that \( d\Omega_k^{N+2}(c)/dc \) is bounded independently of \( c \).

**Definition 3.2** For integer \( m \geq 1 \), we define a polynomial \( S_m(x) \) of degree \( m \) by
\[ S_m(x) = \sum_{n=0}^{m} T_m - n(x) \int_{-1}^{1} T_n(t) dt, \quad m = 1, 2, \ldots, \quad |x| \leq 1, \quad (3.11) \]
and \( S_0(x) = 0 \). Further we define \( S_m(-x) = -S_m(x) \).

**Lemma 3.3** Let \( \Omega_k^{N+2}(c) \) and \( S_m(c) \) be defined by (3.10) and Definition 3.2, respectively. Then we have
\[ \Omega_k^{N+2}(c) = \{ S_{N+2+k}(c) + S_{N-2-k}(c) + S_{N+2-k}(c) + S_{N-2+k}(c) \}/2 \]
\[-S_{N+k}(c) - S_{N-k}(c) + (c^2 - 1) \omega_{N+1}(c) T_k(c) \log((1-c)/(1+c)) \].

**Proof:** It follows from (3.10) and (A.5) that
\[ \Omega_k^{N+2}(c) = \{ I_k^{N+2}(c) + I_k^{N-2}(c) - 2I_k^{N}(c) \}/4 + (c^2 - 1) J_k^{N}(c) \quad (3.12) \]
where \( I_k^{N}(c) \) and \( J_k^{N}(c) \) are given by (2.9) and (2.10), respectively. Since from (A.4) and (3.11) we have
\[ S_m(x) = \frac{1}{4} \int_{-1}^{1} \frac{\omega_{m+1}(t) - \omega_{m+1}(x)}{t - x} dt, \quad m \geq 1, \]
it follows from (A.3) and (2.9) that \( J_k^{N}(x) = 2\{ S_{n+k}(x) + S_{n-k}(x) \} \), which is used in (3.12) to establish Lemma 3.3. □

The following lemma is proven in section 4.1.
Lemma 3.4 Let \( m \) be an integer and define \( G(m) \) by

\[
G(m) = 2m + 2 \log(m - 1) - 1, \quad m = 4, 5, \ldots,
\]

(3.13)

\( G(3) = 9/2 + 1/6, G(2) = 7/2 + 1/6, G(1) = 2 \) and \( G(0) = 1 \). Then for \( S_m(x) \) defined by Definition 3.2 we have with \(-1 \leq x \leq 1\)

\[
\left| \frac{d}{dx} (S_{m+1}(x) - S_{m-1}(x)) \right| \leq 2G(|m|), \quad m = 0, \pm 1, \pm 2, \ldots
\]

(3.14)

On the other hand, in section 4.2 we prove the following lemma, which plays an important role in uniformly bounding the errors of approximate integrals.

Lemma 3.5 For \( \omega_{N+1}(x) \) and \( J_k^N(x) \) defined by (2.2) and by (2.10), respectively, we have

\[
\left| \frac{d}{dx} ((x^2 - 1) J_k^N(x)) \right| \leq 1.8(N + k) + 11, \quad -1 \leq x \leq 1.
\]

(3.15)

From (3.9) and Lemmas 3.3, 3.4 and 3.5 we have the following theorem.

Theorem 3.6 Let \( N \) be an even integer and assume that \( f(z) \) is single-valued and analytic inside and on \( \mathcal{E}_\rho \). Further for \( G(m) \) given by (3.13), define \( H_k(N) \) by

\[
H_k(N) = 1.8(N + k) + 11 + G(N + k + 1) + G(N + k - 1) + G(|N - k + 1|) + G(|N - k - 1|).
\]

Then, the error of the approximate integral \( dq_q(N+2; c)/dc \) is bounded independently of \( c \) by

\[
\left| \frac{d}{dc} q_q(N+2; c) \right| \leq \sum_{k=0}^{\infty} |V_k^{N+2}(f)| H_k(N),
\]

(3.16)

where \( V_k^{N+2}(f) \) is defined by (3.8).

Suppose that \( f(z) \) be a meromorphic function with a finite number of simple poles outside \( \mathcal{E}_\rho \). Then performing the contour integral of \( V_k^{N+2}(f) \) (3.8) yields \( V_k^{N+2}(f) = O(r^{-k-N-2}) \) when \( N \to \infty \), where \( r > 1 \) is a constant [12, 13]. From this fact and (3.16) we see that the error of the approximation (3.6) tends to zero uniformly as \( N \to \infty \).

4 Proofs

4.1 proof of Lemma 3.4

Here we prove Lemma 3.4. Let \( m \) be an integer. Since for \( m < 0 \) from (3.11) we have \( S_{m+1}(x) - S_{m-1}(x) = S_{|m|+1}(x) - S_{|m|-1}(x) \), it suffices to prove (3.14) in the case \( m \geq 0 \). It follows from (3.11) that setting \( M = |m|/2 \),

\[
\frac{d}{dx} S_{m+1}(x) = 2 \sum_{n=0}^{M} \frac{m + 1 - 2n}{1 - 4n^2} U_{m-2n}(x)
\]

(4.1)

\[
= \sum_{n=0}^{M} \left( \frac{m + 2}{2n + 1} - \frac{m}{2n - 1} \right) U_{m-2n}(x),
\]

because \( T'_k(t) = k U_k(t) \) and \( \int_{-1}^{1} T_k(t) dt = 2/(1 - k^2) \) if \( k \) is even, otherwise vanishes. From (4.1) it is easy to verify that \( |dS_1(x)/dx - dS_{-1}(x)/dx| = 2|dS_1(x)/dx| = 2 \), while \( |dS_2(x)/dx - dS_0(x)/dx| = |dS_2(x)/dx| = 2|U_1(x)| \leq 4 \).
Since similarly to (4.6) we have $|(x^2 - 1)T_N(x)T_k(x)\log\{(1 - x)/(1 + x)\}| \leq (1 - x^2)|\log\{(1 - x)/(1 + x)\}| = 0.895486 \cdots < 0.9$, we can easily verify (3.15) by using in (4.4) the inequalities (4.5) and (4.6) and the fact that $|\omega_{N+1}(x)| \leq 2$. 

Using (A.1) in (4.1) yields

$$
\frac{d}{dx}S_{m+1}(x) = 2m \sum_{n=0}^{M-1} \frac{T_m-2n(x)}{2n+1} + 2 \sum_{n=0}^{M} U_{m-2n}(x) + m \frac{U_{m-2M}(x)}{2M + 1},
$$

(4.2)

for $m \geq 0$, where the empty summation should be taken to be zero. It follows from (A.1) and (4.2) that for $m \geq 2$

$$
\frac{d}{dx}S_{m+1}(x) - \frac{d}{dx}S_{m-1}(x)
= 2(m - 2) \sum_{n=1}^{M-1} \left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) T_m-2n(x) + 8 \sum_{n=1}^{M-1} \frac{T_m-2n(x)}{2n+1} + 2(m + 1) T_m(x) + \left(\frac{m + 2}{2M + 1} - \frac{m - 2}{2M - 1}\right) U_{m-2M}(x).
$$

(4.3)

From (4.3) we can verify that $|dS_3(x)/dx - dS_1(x)/dx| = |6T_3(x)+4/3| \leq 7+1/3$ and that $|dS_4(x)/dx - dS_2(x)/dx| = |8T_3(x) + 2U_1(x)/3| \leq 9 + 1/3$. On the other hand, for $m \geq 4$ the inequality (3.14) follows easily since from (4.3) we have

$$
\left|\frac{d}{dx}S_{m+1}(x) - \frac{d}{dx}S_{m-1}(x)\right| \leq 2(m - 2) \left(1 - \frac{1}{2M - 1}\right) + 2(m + 1) + 4 \log(2M - 1) + \left(\frac{m + 2}{2M + 1} - \frac{m - 2}{2M - 1}\right) (m - 2M + 1),
$$

where the relation $\sum_{n=1}^{M-1} 1/(2n + 1) \leq (1/2) \log(2M - 1)$ has been used.

### 4.2 proof of Lemma 3.5

Since $T'_k(x) = kU_{k-1}(x)$ and $U_k(x) + U_{k-2}(x) = 2xU_{k-1}(x)$, it follows from (2.2) and (A.1) that

$$
\frac{d}{dx}\left\{(x^2 - 1)\omega_{N+1}(x)T_k(x)\log\left(\frac{1 - x}{1 + x}\right)\right\}
= \{3x\omega_{N+1}(x) + 2N(x^2 - 1) T_N(x)\} T_k(x) \log\left(\frac{1 - x}{1 + x}\right)
+ \frac{k}{2} \omega_{N+1}(x) \omega_{k+1}(x) \log\left(\frac{1 - x}{1 + x}\right) + 2 \omega_{N+1}(x) T_k(x).
$$

(4.4)

We make use of the fact that $|T_k(x)| \leq 1$ and $|\omega_{N+1}(x)| = 2|\sin \theta \sin N\theta| \leq 2\sqrt{1 - x^2}$, $x = \cos \theta$ and perform numerical computation to show that for $|x| \leq 1$

$$
|x \omega_{N+1}(x) T_k(x) \log\left(\frac{1 - x}{1 + x}\right)| \leq 2 \sqrt{1 - x^2} \left|x \log\left(\frac{1 - x}{1 + x}\right)\right|_{x = \pm 0.89829} \cdots \leq 2 \times 1.155155 \cdots < 2.32.
$$

(4.5)

and

$$
|\omega_{N+1}(x) \omega_{k+1}(x) \log\left(\frac{1 - x}{1 + x}\right)| \leq 4(1 - x^2) \left|x \log\left(\frac{1 - x}{1 + x}\right)\right|_{x = \pm 0.64791} \cdots \leq 4 \times 0.895486 \cdots < 3.59.
$$

(4.6)

Since similarly to (4.6) we have $|(x^2 - 1)T_N(x)T_k(x)\log\{(1 - x)/(1 + x)\}| \leq (1 - x^2)|\log\{(1 - x)/(1 + x)\}| = 0.895486 \cdots < 0.9$, we can easily verify (3.15) by using in (4.4) the inequalities (4.5) and (4.6) and the fact that $|\omega_{N+1}(x)| \leq 2$. 

8
5 Numerical examples

Examples in this section are computed in double precision; the machine precision is $\epsilon = 2.22 \cdots \times 10^{-16}$. Here, we show the numerical results, in particular, for the f.p. integrals with varied values of the pole $c$. Tables 1 and 2 give the approximation errors with the present rule (3.6) for the problems:

$$
\int_{-1}^{1} \frac{1-a^2}{1-2ax+a^2} \cdot \frac{1}{(x-c)^2} \, dx, \quad a = 0.7, 0.8. \tag{5.1}
$$

From Tables 1 and 2 we can see that the present quadrature rule (3.6) gives approximations with the uniform error bound independent of the values of $c$.

Table 1: Errors of the approximations with the number of sample points $N+2 = 65 + 2 = 67$ for the problem \( \int_{-1}^{1} (1-a^2)/(1-2ax+a^2)/(x-c)^2 \, dx \), where $a = 0.7$

<table>
<thead>
<tr>
<th>$c$</th>
<th>exact integral</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.999</td>
<td>-0.5935747038873891E+4</td>
<td>0.14E-7</td>
</tr>
<tr>
<td>0.949</td>
<td>-0.6834948311551221E+2</td>
<td>0.32E-6</td>
</tr>
<tr>
<td>0.849</td>
<td>-0.4543635190689350E+1</td>
<td>0.41E-6</td>
</tr>
<tr>
<td>0.749</td>
<td>0.3352097660293800E+0</td>
<td>0.37E-6</td>
</tr>
<tr>
<td>0.649</td>
<td>0.1028826436467814E+1</td>
<td>0.49E-6</td>
</tr>
<tr>
<td>0.549</td>
<td>0.1042847454119858E+1</td>
<td>0.24E-6</td>
</tr>
</tbody>
</table>

Table 2: Errors of the approximations with the number of sample points $N+2 = 129 + 2 = 131$ for the problem \( \int_{-1}^{1} (1-a^2)/(1-2ax+a^2)/(x-c)^2 \, dx \), where $a = 0.8$

<table>
<thead>
<tr>
<th>$c$</th>
<th>exact integral</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.999</td>
<td>-0.9725245010078143E+4</td>
<td>0.24E-9</td>
</tr>
<tr>
<td>0.949</td>
<td>-0.3030594600624694E+2</td>
<td>0.14E-8</td>
</tr>
<tr>
<td>0.849</td>
<td>0.4565803638633153E+1</td>
<td>0.82E-8</td>
</tr>
<tr>
<td>0.749</td>
<td>0.3531712396524727E+1</td>
<td>0.97E-8</td>
</tr>
<tr>
<td>0.649</td>
<td>0.2463752827083085E+1</td>
<td>0.61E-8</td>
</tr>
<tr>
<td>0.549</td>
<td>0.1785319047569635E+1</td>
<td>0.76E-8</td>
</tr>
</tbody>
</table>

approximations with the uniform error bound independent of the values of $c$.

A Appendix A

Here we collect some relations on the Chebyshev polynomials useful in this paper. The relations [21, pp.5-9]

\[
2T_n(x)T_m(x) = T_{n+m}(x) + T_{|n-m|}(x), \quad n \geq 0, \quad m \geq 0, \\
U_k(x) - U_{k-2}(x) = 2T_k(x), \quad k \geq 2, \tag{A.1}
\]

\[
T_{k+1}(x) - T_{k+1}(c) = 2(x-c) \sum_{n=0}^{k} U_{k-n}(c) T_n(x) \\
= 2(x-c) \sum_{n=0}^{k} U_{k-n}(x) T_n(c) \quad k \geq 0, \tag{A.2}
\]

gives

\[
2T_n(x) \omega_{N+1}(x) = \omega_{N+n+1}(x) + \text{sign}(N-n) \omega_{|N-n|+1}(x), \quad n \geq 0, \quad N \geq 1, \tag{A.3}
\]
\[ \omega_{k+1}(x) - \omega_{k+1}(c) = 4(x-c) \sum_{n=0}^{k} \alpha_n T_{k-n}(c) T_n(x), \quad (A.4) \]

\[ 4(x^2 - 1) \omega_{N+1}(x) = \omega_{N+3}(x) + \omega_{N-1}(x) - 2 \omega_{N+1}(x), \quad (A.5) \]

since \(2(x^2 - 1) = T_2(x) - 1\), where \(\text{sign}(k) = 1\) if \(k > 0\), otherwise \(\text{sign}(k) = -1\) while \(\text{sign}(0) = 0\). For (A.2) see Elliott [7].

References


